MISSED OPPORTUNITIES: INSTANCES FROM GEOMETRY LESSONS

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We present instances from two high school Geometry lessons where the teacher is faced with contingent (Rowland & Zazkis, 2013) situations. We propose a framework to analyse the responses of the teacher in each of these scenarios to understand opportunities taken, built on and opportunities missed. Then, we propose some suggestive activities that could be taken up for these scenarios consistent with the mathematical modes of inquiry. The framework is aimed at providing insights to teachers to reflect on their teaching learning practices and help in their development of professional noticing (Jacobs, Lamb, & Philipp, 2010).

WHY INDIAN CLASSROOMS AND MATHEMATICAL INQUIRY?

Indian classrooms are typically large sized. With an average class strength of around 50 students or more, it becomes often difficult for teachers to address students' diverse responses. However, if the teacher does not pick at least some of these responses to probe their thinking further, such a practice cannot lead to meaningful facilitation. 'Professional noticing of children's mathematical thinking' entails an integrated teaching move that comprises of attending to mathematical strategies in students' responses, interpreting these details against research on children's mathematical development and deciding on how to respond (Jacobs et al., 2010). The discourse on 'missed opportunities' (Rowland & Zazkis, 2013) seeks to attend to the opposite of planning, to situations that are unplanned and require an act of improvisation by the teacher. The contingent situations, generally arising out of students' responses in classrooms, are excluded from the teacher's lesson image (Schoenfeld, 1998). Rowland and Zazkis further argue that contingent situations provide opportunities for educators to demonstrate a commitment towards the modes of inquiry in mathematics. That apart, such situations may also induce anxiety in teachers due to "uncertainty about the sufficiency of one's subject matter knowledge" (Rowland, Huckstep, & Thwaites, 2005, p. 263). Research suggests that a better understanding or experience with foundation, transformation and connection (Rowland et al., 2005) aspects of mathematical knowledge would better dispose teachers to be able to meaningfully deal with such situations and drive mathematical modes of inquiry. Rowland identifies two aspects to responding to contingent situations: readiness to respond to children's ideas and preparedness to deviate from a set out agenda. In this paper, we will examine two contingent classroom situations from the classroom conversations of Grade 9 students from a government-run school in Dhamtari district of Chhattisgarh state in India. The aim of the analysis is to not identify gaps in teaching or teacher knowledge but to trace trajectories of teacher-support for furthering possible explorations in school mathematics lessons.

THEORETICAL ORIENTATION

Previous research has shown that when pursued contingent situations have the potential of providing some interesting and fruitful learning opportunities (Rowland et al., 2005). The capacity to make fruitful use of such situations by teachers is dependent on their knowledge of the mathematical potential in the contingent situation and a commitment to mathematical enquiry, as illustrated through the examples of Laura and Bishop's (Rowland & Zazkis, 2013) stories. Expanding on Rowland's framework for this analysis, we propose following framework to analyse the teacher's responses to contingent situations:

Case 1: Opportunity underestimated or not understood
Case 2: Opportunity understood but ignored due to conscious choice or mathematical potential not recognized
Case 3: Opportunity realised but unable to build on the mathematical potential
Case 4: Opportunity realised and successfully built on the mathematical potential

The analysis follows with a suggestive grade appropriate exploration that educators can take up as follow-up of such contingent situations, which is dependent on teacher's ability to notice such situations and follow up on them through designed experiences. The use of Dynamic Geometry Environments (DGEs) for guided exploratory activities (Zbiek, Heid, & Blume, 2010) have been developed and analysed through the lens of expressive and exploratory nature of such exercises. We have used van Hiele theory of students' geometric reasoning development as the underlying structure for elaborating on the levels and types of these dynamic activities (Manizade & Mason, 2010).

METHODOLOGY

Non-participant classroom observations were done by the researchers as part of the Connected Learning Initiative (CLIx) programme (www.clix.tiss.edu), an ICT based educational intervention run by TISS, Mumbai, being implemented in select secondary schools in four states of India. The study focused on lessons on Geometric Reasoning taught to Grade 9 students. The observers (authors and colleagues) took running notes sitting at the back of the classroom and audio-recorded the classroom proceedings for triangulation. These observations were preceded and followed up by conversations with the teachers about their teaching plans and reflections on teaching. Post observation phase, contingent situations were identified from these notes and analysed as per the framework proposed above.

For the creation of guided exploratory DGE (GeoGebra) based experiences, expressive (Sherman, 2010) activities have been created and a suggestive approach to these mathematical inquiries has been laid down in order to provide concrete strategies for teacher facilitation. GeoGebra, a free DGE has been used for these activities in line with the existence of the ICT infrastructure and facilities available in the concerned school.

CLASSROOM SCENARIOS

From the point of view of researchers, it is always possible to point out to aspects of classroom, instruction, assessment, resource materials and classroom processes that may theoretically not be invoked in their ideal manifestations. Instances like not seeing value in discussing the 'axis of rotation' while talking about rotation of 2D shapes, ignoring a students' response that 'all sides of a parallelogram are parallel', not discussing all possible types of trapeziums while talking about right angles contained in a trapezium, etc., can be looked at from the perspectives of missed opportunities as examples of case 1 of the framework, wherein the teacher doesn't notice the contingency in a mathematical situation. It is a matter of active pedagogical choice for teachers to respond to or build on the incorrect utterances and doubts raised in class. The ideas of teacher's discretion and autonomy in making pedagogical choices are important aspects of the profession. In the absence of a robust system for teacher education in the country, can we always pose such an argument in the favour of educators? This section investigates two instances of teacher-student interactions.

Scenario 1

The teacher had been teaching the chapter on quadrilaterals for the past 4-5 mathematics lessons. This class started with a recap of the previous topic, types of quadrilaterals. Towards the completion of the recapitulation, the teacher asked students to frame questions to distinguish between quadrilaterals, based on their properties. Students were divided into two subgroups as boys and girls, and each sub group had to ask questions one after the other. The other group was supposed to answer the posed question and the teacher intervened and moderated turns. The following extract is taken from one such teacher-student interaction of about 7-8 minutes.

1	S_b	Why is none of the angles of a kite a right angle?
2	T	Who will answer this question? Now, try this in the matchstick shapes[1] that you have made.
3	T	Try making a right angle.
4	S_{g1}	Sir, a kite is being formed.
5	S_{g2}	It has become a trapezium.
6	T	No, a scalene quadrilateral is formed. Show this on the black board.

S_{g2} brings the manipulative and the teacher copies the shape on the black board.

7	T	Which quadrilateral is formed?
8	S_{g2}	Scalene quadrilateral
9	T	This has to be made into an angle of 90^0. But if we do that, the shape is deformed and no longer will remain a kite. Good question, sit down.

T - Teacher, $\mathbf{S_{b1}}$ - first boy student, $\mathbf{S_{g1}}$ - first girl student

1 – The students had done an activity based on making different shapes (particularly quadrilaterals) using match sticks and cycle valve tubes (a low-cost teaching learning material).

During post-class interaction, the teacher agreed that this possibility had never been explored by him before. This question was clearly not in the lesson image of the teacher, hence a contingency. The teacher thought that the question was dealt correctly using the concrete manipulative. The concerned teacher had even named the most general kind of quadrilateral as scalene quadrilaterals. He realised the mathematical potential in the question and used his transformative knowledge (Rowland et al., 2005) by asking a student to come and demonstrate the construction of a right angle in a kite, through the concrete manipulation of sticks joined through nuts and bolts, and threads hung from these bolts. However, he was not able to drive it to meaningful exploration of the possibilities. The process applied lacked rigor in bringing out the nature of the problem, hence belongs to case three category of the framework. Here, the teacher's noticing goes through the steps of attending, interpreting and responding, yet doesn't do justice to the problem, probably due to dependency on physical construction of right angle in a kite, a case of inductive method, or verification. This problem is at vanHiele level 3 (abstraction) (Manizade & Mason, 2010) because students engage with the definition of kite and the existence of right angle(s) at least requires informal arguments to justify. There is no attempt to generalise this notion for the entire set of kites, a problem pitched at level 4 (deduction).

Scenario 2

Another such teacher-student interaction from the same lesson is presented below:

1	T	Now, tell what a parallelogram is.
2	S_{g1}	If we change all the four angles of a rectangle, it becomes a parallelogram.
3	T	Any other answers?
4	S_{b1}	All of its angles are 90^0.

T repeats the sentence said by the student in a dissatisfied tone giving negative reinforcement.

5	S_{b2}	The opposite sides are equal.
6	T	This property is held by rectangle, square and rhombus. How can we tell then?
7	S_{g2}	That which has opposite lines equal and parallel.
8	S_{g3}	Whose difference of the lengths of the sides are equal.
9	T	Means you are saying that the opposite sides are equal. Think some more.
10	S_{g4}	A quadrilateral that has its opposite sides parallel but angles not equal to a right angle.
11	T	Yes, very good.

This situation also concerns vanHiele level 3 (abstraction) as it involves perceiving relationships between properties, creating meaningful definitions, and justifying through arguments. It is also concerned with level 4 (deduction) due to the inclusion of ideas of necessary and sufficient conditions for the construction of a parallelogram. The teacher appreciated response from S_{g4} and ended the conversation legitimizing only her definition of parallelograms as 'correct'. It appears that the teacher was looking for a specific definition of parallelograms as he ignored the previously related responses by S_{g1}, S_{b1}, S_{b2}, S_{g2}, and S_{g3}, without meaningfully engaging with any of them. The teacher does not attend to the mathematical details (Jacobs et al., 2010) in these responses. Some of the equivalent definitions of parallelograms could have been taken up as subjects for exploration before moving onto the next question. Hence, it is a case of missed opportunity due to non-realization of potential for mathematical inquiry, hence, case two. However, in a stricter sense, this situation does not qualify as contingent, as the teacher thought that all the responses fell within his observations space (Rowland & Zazkis, 2013).

S_{g1}'s response appears to be shaped by the students' experience of manipulating different geometric shapes using matchsticks (a learning aid). With this material, pressing the opposite angles of the rectangle always yields a parallelogram. Theoretically, the angles of rectangle can be changed in many different ways and only a particular combination of such manipulation yields a parallelogram. Different such combinations of ways of manipulation could have been explored here for building deeper understanding. The response of S_{b1} (all right angles) again indicates the use of specific types of parallelograms (rectangles), which was neither challenged nor built upon by the teacher. The definitions provided by S_{b2} and S_{g2} are mathematically consistent and form a case worthy of acknowledgement and taking up for discussion by the teacher. The emphasis on the acknowledgement and exploration of the multiple definitions resonates with the view, "Saving school mathematics from the tyranny of one correct answer" (National Council for Educational Research and Training [NCERT], 2006, p. 6). Student S_{g3}'s response is complicated and could have been understood further only through probing questions, hence a missed opportunity. The teacher oversimplifies the statement and reduces it to the response given by student S_{b2}. Student S_{g4} quotes the standard definition provided in textbooks but she adds this additional condition about each angle not being equal to the right angle. This is a Partition definition (De Villiers, 1994) that excludes rectangles and squares as special cases of parallelograms. Although some of these definitions could only be proved through formal deduction using the concept of 'congruence of triangles' (a topic introduced later in the curriculum), nevertheless, these can provide students avenues to verify and explore necessary and sufficient conditions, to understand mathematical invariances (class of quadrilaterals, parallelograms, in this case) in mathematics.

SUGGESTIVE EXPLORATIONS

We have tried to build on the analysis of teacher responses to propose possible explorations aimed at reorganising (Sherman, 2010) children's thought processes about relationships between properties of the above discussed quadrilaterals and the necessary and sufficient conditions involved, by using GeoGebra. Although given as procedural steps, these are indicative approaches that can be followed for verifications, understanding need for mathematical proof and developing it. We noted that students needed to be given opportunity to construct own steps of construction using GeoGebra tools before they were guided through ready-made procedural steps. Such an approach was better disposed towards the expressive nature (Sherman, 2010) of students' mathematical goals. The activities proposed are of type 2 (Developing Abstraction) and type 3 (Developing Deduction) (Manizade & Mason, 2010).

Scenario 1: Exploration of Kites

Kites can be mathematically defined as 'quadrilaterals with two pairs of (disjoint sets) equal adjacent sides. This partitive definition does not include rhombuses and squares as special cases of kites. Another possible definition can be 'quadrilaterals with two pairs of equal adjacent sides. As a consequence of this inclusive definition, rhombuses and squares can be included to be considered as special cases of kites. Hence, squares can be considered as cases of kites having four right angles simultaneously, or rhombuses as kites having all equal sides. However, the partitive definition can be used to deal with the other two cases where a kite has right angles. Kites have one pair of opposite angles equal and another pair of opposite angles that are unequal. These properties give rise to two possibilities – first, the pair of equal opposite angles can simultaneously be equal to right angles and second, one of the unequal pairs of opposite angles can be a right angle.

To guide students towards exploration of these possibilities, one can start by taking rhombus as the limiting case. We can either stretch or push inwards one pair of adjacent sides of the rhombus, along the diagonal that is not bisected by the other diagonal using sliders in GeoGebra. This process of manipulation transforms (in abstract) or manipulates (with a concrete model) the rhombus into kites. In this process of manipulation, one can use protractors with concrete models to arrive at kites that have the equal opposite angles as right angles. A dynamic software environment like GeoGebra can also provide students an opportunity for such explorations. The snapshots of two cases of kites having right angles is shown in Figure 1.

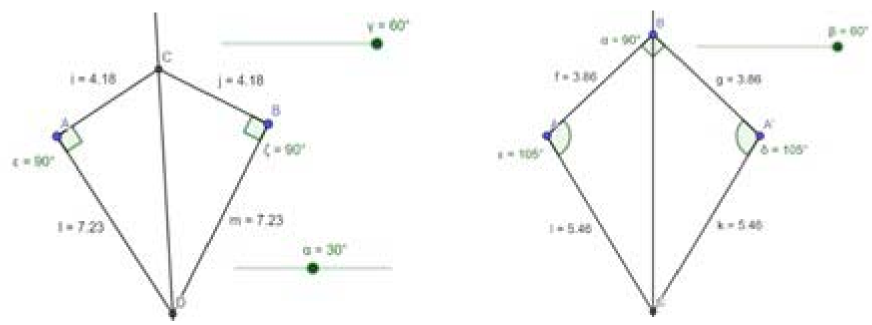

Figure 1: Right angles(s) in Convex Kite

A dynamic environment can help students verify other properties and invariances resulting out of such restrictions. The first image in Figure 1 is the case of a 'right kite' formed from two congruent right triangles as can be proven using the RHS criteria of triangle congruence. This is also a cyclic quadrilateral, since both pairs of opposite angles are supplementary. A square can be understood to be a right kite with equal diagonals. A right kite can never be a concave quadrilateral as the reflex angle and the two right angles will sum to more than 360^0, degenerating the quadrilateral itself. The second image in Figure 1 is the case of a kite having one right angle. This kite can be manipulated into a concave quadrilateral as shown in Figure 2.

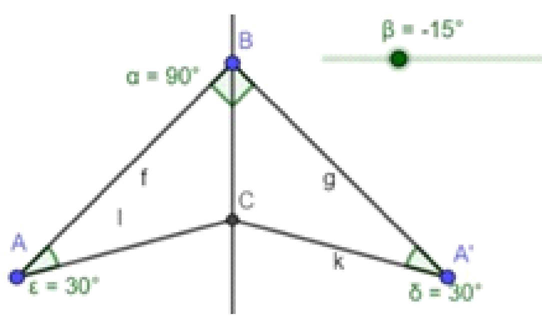

Figure 2: Right angle in Concave Kite

Multiple definitions of parallelograms exist depending on the type of restrictions that are imposed on quadrilaterals. The beauty of the same invariance resulting from the imposition of different restrictions can provide students with interesting opportunities for mathematical exploration. Eight possible definitions of parallelograms have been discussed in this section, out of which three (Definitions 1, 3, and 6) were mentioned by students in the classroom scenario. The definitions and steps for exploration, along with relevant GeoGebra snapshots, have been listed below one by one.

Quadrilaterals with both pairs of opposite sides parallel.

Draw two arbitrary intersecting lines *AB* and *AC* using the Line tool as shown in Figure 3. Now, draw a line through point *C* which is parallel to line *AB* and another line through *B* which is parallel to line *AC*, using the Parallel Line tool. Mark the point of intersection of these two lines through points *B* and *C* as *D* using the Intersect tool. Through a *drag test*, the quadrilateral *ABDC* can be verified to be a parallelogram through its properties.

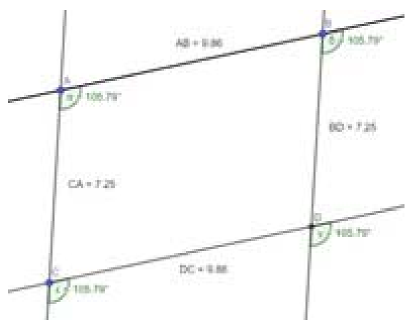

Figure 3: Definition 1

Quadrilaterals whose diagonals divide them into two congruent triangles.

Draw a triangle *EFG* using the polygon tool. Find the midpoint *H* of the side *EG* using the midpoint tool. *EFG* using the polygon tool. Rotate triangle *EFG* about point *H* by 180^0 to get triangle *E'F'G'* using the Rotate around Point tool. Quadrilateral *EFGF'* thus formed is a parallelogram as shown in Figure 4.

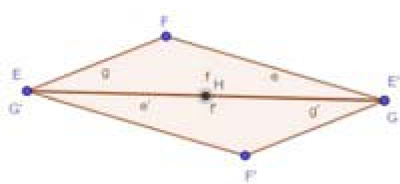

Figure 4: Definition 2

Quadrilaterals with both pairs of opposite sides equal.

Draw two intersecting lines *AB* and *AC* using the Line tool. Now, draw a circle *c* with centre as *C* and radius as *AB* and another circle *d* with centre *B* and radius as *AC*, using the Circle with Centre tool. Mark the point of intersection of the circles *c* and *d* as *D* using the Intersect tool. Join line segments *BD* and *DC* using the Segment tool. Quadrilateral *ABDC* is a parallelogram as shown in Figure 5.

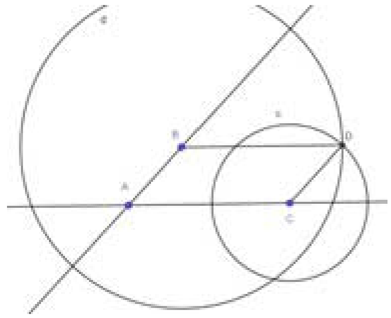

Figure 5: Definition 3

Quadrilaterals with both pairs of opposite angles equal.

Create two angle sliders

α and γ with range 0^0 to 180^0 using the tool. Draw an angle *ABA'* with size as α using the tool. Take an arbitrary point *C* on side *BA'*. Create two angles *BCB'* and BAB'_1 with size γ using the tool. Mark the intersection point of lines *CB'* and AB'_1 as *D* using the tool. Mark the angle *CDA* using the tool. Now, use the *drag test* to see for what values of α and γ does the quadrilateral *BCDA* become a parallelogram as shown in Figure 6.

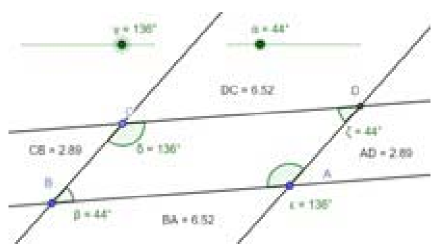

Figure 6: Definition 4

Quadrilaterals whose diagonals bisect each other.

Draw a line segment AB using the Segment tool, and find its midpoint *C* using the Midpoint or Centre tool. Then, draw a line segment *CD* of an arbitrary length using the Segment tool. Rotate line segment *CB* by 180^0 using the Rotate around Point tool and join the four end points *ADBD'* to get a parallelogram as shown in Figure 7.

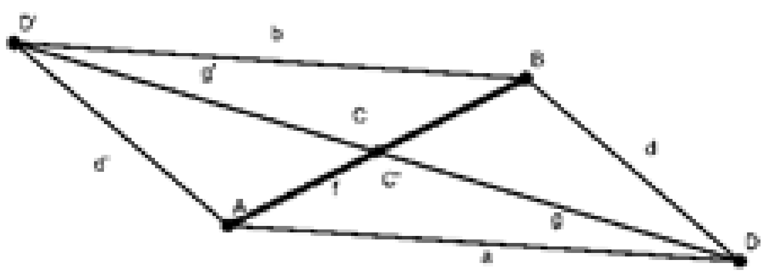

Figure 7: Definition 5

Quadrilaterals with one pair of opposite sides equal and parallel.

Create a number slider *a* with range 0 to 5 units using the Slider tool. Draw two line-segments *AB* and *CD* of length *a* using the Segment with Given Length tool. Draw a line through point *D* parallel to segment *AB*, using the Parallel Line tool. Drag point *D* manually to make the segment *CD* become parallel to segment *AB*. Join points *ABDC* using the tool to get a parallelogram as shown in Figure 8.

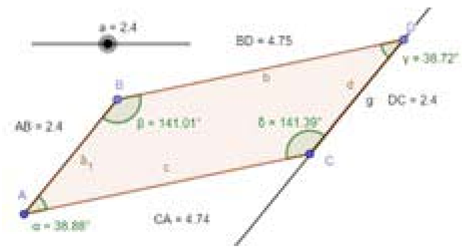

Figure 8: Definition 6

Quadrilaterals with a pair of opposite sides parallel and a pair of opposite angles equal.

Draw a line *AB* using the Line tool. Draw a line *g* parallel to *AB* that passes through an arbitrary point *C* using the Parallel Line tool. Join points *A* and *C* using the Segment tool to make a transversal to the parallel sides. Mark angle *CAB* using the Angle tool as *á*. Choose an arbitrary point *D* on line *g*. Draw an angle *CDC'* equal to á in the clockwise direction, using the Angle with Given Size tool. Draw line *DC'* using the Line tool. Mark point *E* as the intersection point of lines *AB* and *DC'* using the Intersect tool. Quadrilateral *ACDE* thus formed is a parallelogram as shown in Figure 9.

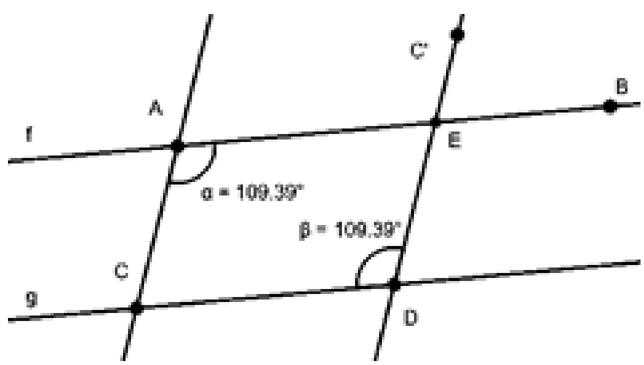

Figure 9: Definition 7

Quadrilaterals with one pair of opposite sides equal and one pair of opposite angles equal.

This definition seems to flow naturally from some of the above explored statements. A preliminary exploration in GeoGebra also seems to verify/confirm this conjecture. However, such a parallelogram cannot be uniquely determined. To falsify this statement, we require to construct just one counter-example, as shown in Figure 10. *BQ* and *BD* are radii of the same circle. Triangle *ABQ* has been rotated about point *B* such that *BQ* coincides with *BE*. This triangle is then reflected along the segment *BE* and the resulting triangle is again flipped about the perpendicular bisector of *BE*. Quadrilateral *ABCE* thus formed has equal opposite sides *AB* and *CE* and equal opposite angles *A* and *C*, yet it is not a parallelogram. Hence, this statement is falsified and rejected as an invalid definition.

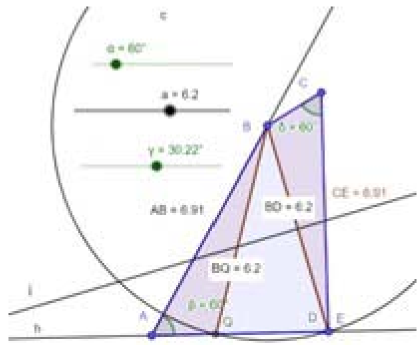

Figure 10: Definition 8

LIMITATIONS

The student participants could not be interviewed for this research. There are times when teachers took an informed choice of postponing the elaboration of a concept during a lesson. The two instances discussed in this paper have been taken from a classroom quiz activity wherein the scope of deviations from the set-out agenda is limited by design. The teacher's active facilitation made the contingent situations possible. The analysis may not only be seen as a critique but as inputs for considerations for teacher support, especially in teachers' noticing of geometric reasoning lessons. Moreover, the discussion on the framework remains incomplete in the absence of a fourth case of the framework.

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