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METAPHOR-EQUIPPED TEACHING OF LINEAR ALGEBRA

Praveen Chhikara
Mathematical Community Trust, New Delhi, India
chhikara.praveen18@gmail.com

The article presents a study about using metaphors or analogies as a tool to illustrate some introductory concepts of linear algebra. It conducts a brief survey in an attempt to explore the effects of metaphors in teaching linear algebra and with that it argues in favor of equipping the teaching of formal mathematics with metaphors used in real life. At the same time the article acknowledges the presence of potential pitfalls of metaphors and with illustrations, suggests ways to minimize them using the FAR guide.

INTRODUCTION & LITERATURE REVIEW

"New ideas come from old ideas that are revisited, reviewed, extended, and connected" (Maher, 2010). If an advanced topic in a mathematics classroom is connected with previous knowledge, it is likely to ease the problems of a lot of students, struggling with abstract concepts of the subject (Gentner, Holyoak & Kokinov, 2001). The new knowledge should be constructed from experience and prior knowledge, otherwise it causes fear of mathematics among children (NCERT, 2005; NCTM, 2000).

To make connections with a previous knowledge, teachers adopt an instructional approach of employing nonmath analogies or metaphors (Sarina & Namukasa, 2010; Richland, Holyoak & Stigler, 2004). Using an analogy or a metaphor means understanding a situation in terms of the other. Many times, it offers linking or comparing the abstractions in mathematics with simpler and more familiar ideas. "Many of the most fundamental mathematical ideas are inherently metaphorical in nature" (Lakoff & Núñez, 2000). Sfard (2008) describes metaphors as generators of new discourses. Thus, the technique of associating metaphors should not be seen as artificial.

Frant, Acevedo, and Font (2006) examine how teachers use metaphors while teaching graph functions in a mathematics classroom. Richland, Holyoak, and Stigler (2004) identify analogies in a random sample of eighth-grade mathematics classrooms. They analyze patterns of teacher-student participation, analogy source and target constructions, and contexts for analogy constructions. There is a paucity of research examining the influence of metaphors in teaching and learning linear algebra. Adiredja and Zandieh (2017) reveal how eight women of color understand the concept of basis in linear algebra using intuitive ideas from their daily lives. Sweeney (2012) offers a travelling metaphor and its impact on reasoning in linear algebra.

The present article investigates how metaphors or analogies can be used as a device to illustrate some basic concepts of linear algebra. It carries out a survey which aspires to find the effects of metaphors in teaching linear algebra and with that it argues in favor of equipping the teaching of formal mathematics with metaphors used in real life. Further, it attempts to underscore the significance of the following questions in relation to linear algebra: Do metaphors generate new knowledge or are they just excellent communication tools (Harrison & Treagust, 2006)? How much control does a teacher have over his/her metaphor usage? Does the teacher know the positive/"negative" effects of metaphors in the negotiation of meaning (Richland, Holyoak & Stigler, 2004)? How to keep down the "negative" impacts of metaphors, if any? Can a *single* analogy suffice for a concept? These questions come to the surface substantially when the survey finds some alternative conceptions of the metaphors among the students. The article aims to consider some techniques to reduce the "negative" effects of metaphors while dealing with these questions.

THEORETICAL FRAMEWORK

An abstract unknown domain is conceptualized in terms of some concrete familiar domain. The former is called the target domain, and the latter is called the source domain. We are familiar with the source domain but not so with the target domain. The source domain serves as a source of knowledge about the target domain, that we want to investigate. A metaphor "*A* is like *B*" is a mapping $m: B \to A$ from *B* to *A*, where *B* is a source domain and *A* is a target domain (Lakoff & Núñez, 2000; Gentner, 1983). The metaphor *m* sends entities in the conceptual domain *B* to corresponding entities in the other conceptual domain *A*. The inferential structure of the familiar domain guides students to "argue" about the unfamiliar domain.

While teaching through metaphors, students' personal construction of meaning may differ from the teacher's intended knowledge (Harrison & Treagust, 2006). To minimize these "disadvantages" of a metaphor, Treagust, Harrison, and Venville (1998) offer a set of steps: the Focus-Action-Reflection (FAR) guide. The first stage of Focus involves a pre-lesson activity of thinking if a concept to-be-taught is difficult for the students, and then an analogy is generated by the teacher. After that, in class, the degrees of likeness and unlikeness of the analogy with the concept is checked in the second stage of Action. The last stage of Reflection involves a post-lesson activity of modifications or changes in the analogy. The modifications may come in terms of replacement of the analogy, multiple analogies for a single concept, or else (Harrison & Treagust, 2006). The FAR guide goes through iterations to evaluate and qualify the metaphors so that they serve well in teaching and learning.

METAPHORS

This section proposes metaphors for some introductory concepts of linear algebra. The following metaphors may seem naïve, crude and simple for someone, but are likely to be very helpful. For definitions, please see some standard text of linear algebra (Friedberg, Insel & Spence, 2003).

1. The linear combination of vectors is like a mixture of things.
2. The span of a set in a vector space is like a collection of all mixtures of things. If $\alpha \in$ span *S*, then α is a mixture of some vectors in *S* or α can be made from vectors in *S*.
3. A set *S* is called a spanning set (SS) of a vector space *V*, if it spans the entire *V*. In other words, if *S*

is a spanning set of V then every vector in V can be made from vectors in S, or we can say the set S can make any vector of V, or S has shortage of nothing. A spanning set has no shortage, so it is also like a person who is rich.

4. A set S in a vector space V is linearly dependent (LD), if and only if there is some vector v in S which is a linear combination of other vectors in S, or there is a vector v in S which can be made from other vectors in S. If this v were not in S, still it could have been made from other vectors in S, so let us call such vector v as redundant vector in S. An LD set is like an object with a redundant thing.
5. A linearly independent (LI) set S in a vector space V is like an object in which nothing is redundant, or it is like a person who is poor.[1]
6. Suppose a child gets pocket money from his/her parents. The child thinks that there should not be shortage in the pocket money and simultaneously the parents think they should not give any redundant money. To make both the child and the parents happy, there should be neither shortage nor redundancy in the pocket money. In other words, the money (regarded as vectors, for a while) should be an SS as well as an LI set. In such an ideal situation, the set is called a basis. A set B is said to be a basis of V, if it is linearly independent and a spanning set of V. A basis is like an object with no shortage, and has no redundant thing.

The following table recapitulates the above metaphors, which were mainly told to the students.

Source Domain	Target Domain
Mixture of vectors	Linear combination of vectors
Collection of all mixtures of things in S	Span S
S has shortage of nothing (rich)	S is a spanning set
At least one redundant vector in S	S is linearly dependent
No vector in S is a redundant vector (poor)	S is linearly independent
An ideal situation (no shortage & no redundancy)	Basis

Table 1: A recap of metaphors stated above

METHOD

The survey participants were 76 students who had just passed grade 12 final exams from various schools in Bihar, India, who were called together with the help of a teacher. They were introduced by the author (who met them for the first time) to some introductory concepts of linear algebra not by formal definition, but via the metaphors given in the previous section, in five half-hour classes. In initial classes, some hands-on sheets were discussed to make them familiar with these concepts. Then, in the final class, they were surveyed through a series of questions. The Fill-in-the-Blank and True-and-False questions first motivated them to write a *source domain statement*, in their native language Hindi, about the source domain, and next to it, write an analogous *target domain statement* about the target domain. After the questionnaire, the students

[1] (Some of my known persons rose objection saying poor among students dishonorable). I still want to keep it for quantitative purpose only, at least for the time being.

were divided in four equal groups and informal verbal discussions followed separately with each group. In this way, the survey tested if the students were capable of making intended conjectures about the target domain, without having been introduced to formal definitions. Although linear algebra is an undergraduate course, the survey participants were students, all of whom had just graduated grade 12, and none of whom had studied linear algebra earlier. To be clear, the purpose of the study was not to introduce the topics to school students. Showing the effects of metaphors-alone, the study was conducted on students, who had not studied linear algebra earlier, and the "positive outcomes" of the study convince us that if undergraduates are introduced to formal definitions accompanied by analogies or metaphors, it would be likely to help make learning more sensible. The responses of the students which are subjective in nature were analyzed collectively (on request) by two professors to determine the efficacy of the method, and other possible advancements, along with identifying "faults" in it. The application of the FAR guide is urged to fix those faults in the later sections.

TASKS

A questionnaire (originally in Hindi-cum-English language), that was given to the students, is provided in this section. The blanks, which were empty originally, are filled below with the desired answers. The initial blanks motivated them to write a statement from a source domain, then the final blank in each question is about the intended conjecture, that the participants were expected to fill about the target domain. There are some assumptions mentioned below, which were thought not important to be told to the participants during the survey.

QUESTION 1

Let $X \subseteq Y$ (see Figure 1). Suppose X has a redundant thing (vector) u. Fill the blanks below.

a) X has a redundant vector $u \Rightarrow$ X is LD.

b) Does Y have a redundant vector? Yes.

c) What type of set Y is? Y is LD.

If a "smaller" set has a redundant thing, then a "bigger set" also has a redundant thing (referring bigger set to superset, and smaller set to subset) (*source domain statement*).

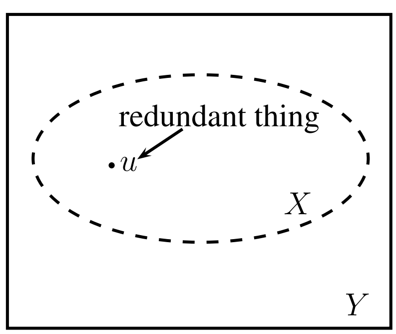

Figure 1

Intended Conjecture: A superset of a linearly dependent set is linearly dependent (*target domain statement*).

QUESTION 2

Let $X \subseteq Y$ (see Figure 2). Suppose Y has no redundant thing.

What statement would you like to propose? Fill the blanks below.

If a "bigger" set has no redundant thing, then a smaller set also has no redundant thing (*source domain statement*).

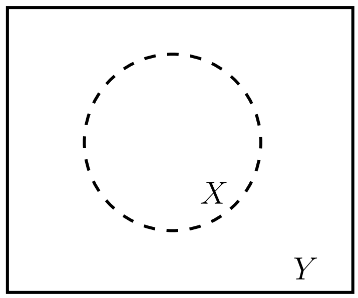

Figure 2

Intended Conjecture: A subset of a linearly independent set is linearly independent (*target domain statement*).

QUESTION 3

Let S be a nonempty subset of a vector space V (see figure 3). Suppose S has no redundancy. The figure shows the "region" of span S, encircled by the oval bold boundary, containing S The rectangular region shows V. Fill the blanks below.

(a) What type of set S is? S is linearly independent.
(b) Which vectors can be constructed from the vectors of S? Dark the region containing those vectors.
(c) Seeing the figure, fill the blank ahead such that the set $S \cup \{u\}$ has no redundant vector. What type of set this is? $S \cup \{u\}$ is also linearly independent.
(d) If S has no redundant thing, and if α is not an element of span S, then $S \cup \{\alpha\}$ has no redundant thing (fill the blank with details about the whereabouts of α) (*source domain statement*).

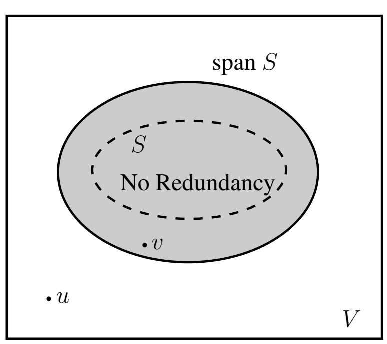

Figure 3

Intended Conjecture: If S is linearly independent and $\alpha \notin$ span S, then S $\cup$ $\{\alpha\}$ is also linearly independent (target domain statement).

QUESTION 4

Fill the blank below:-

Cost of a Rich person's total property ≥ Cost of a poor person's total property (although there is a technically error, see assumptions below) (source domain statement).

Assumptions: There are two assumptions. Firstly, the cardinality of a spanning set and a linearly independent set can be same, for instance, a basis of the space, but we do not find much harm not laying stress on it among the participants, because this model is de facto for undergraduates, who would complete the argument that card (an SS) ≥ card (an LI set), once they know formal definitions. Secondly, we are working in a finite-dimensional vector space, although card (an SS) ≥ card (an LI set) holds in every vector space, once Zorn's lemma is invoked (Bourbaki, 1974).

Intended Conjecture: The cardinality of a spanning set > The cardinality of a linearly independent set (target domain statement).

QUESTION 5

The top box (set) in the Figure 4 is an SS and an LD set. This set must have some redundant vector, which is thrown out to get a smaller set, shown just below the top. The smaller set must be an SS because throwing out a redundant thing, would not lead to any shortage. Suppose it is LD, as shown. The same thing we do with it as we did with the set at the top. It is repeated until we get a set first time with no redundancy. This set represents an ideal situation. Below it, all are LI sets, and above it, all are SS.

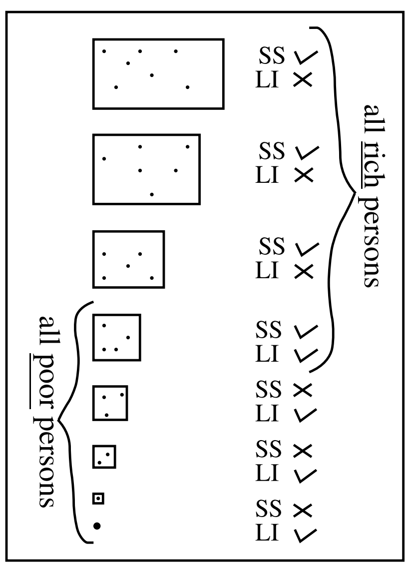

Figure 4

(a) Fill the two blanks (aside the brackets) in the figure about the type of persons above and below the ideal situation (basis).

(b) What type of rich is the ideal-situation-box with respect to other rich? Fill the blank ahead. The rich person in the ideal situation is less rich than other rich (although it is technically wrong, see assumptions below) (*source domain statement*).

Assumptions: There are two assumptions. Firstly, for the time being, we consider no difference between a minimal element and a minimum element of a poset. Secondly, we assume Zorn's lemma, so that we can say an arbitrary vector space has a basis.

Intended Conjecture: A minimum (or minimal) spanning set is a basis (although it is technically wrong as there need not be a unique minimal spanning set, see assumptions above) (*target domain statement*).

FINDINGS

The responses of the survey participants were evaluated to see if they make some progress towards the intended conjectures. Their responses might contain naïve words, they may not be able to write the conjecture in a complete formal code, they may make grammatical errors, the evaluation ignores them all, and seeks to address if the responders intuitive answers are satisfactory, partly satisfactory, wrong, or unintended replies.

Even though there are a significant number of responders who could not reply satisfactorily for each question, the large number of responders who replied satisfactorily seems to show the magic of metaphors-alone (see table 2). One is likely to believe, if undergraduates are introduced to formal definitions equipped with these metaphors, it makes a big impact on their understanding.

Q. No.	Satisfactorily attempted responses	Partly satisfactory attempted responses	Wrong responses	Un-attempted or Undesired responses
1	58	2	4	12
2	57	2	2	15
3	59	11	2	4
4	61	2	2	11
5	54	1	0	21

Table 2: Evaluated responses to the questions

As per FAR guide's second step of Action, wonderful replies were collected from the participants, which are worth mentioning. For the fourth question, two responders stepped even ahead of the intended reply, adding that the cardinality of a spanning set may be equal to that of a linearly independent set. The last question's responses are relatively more impressive; a basis is a needy rich, a basis is less rich than the other rich (technically it is not correct, because there are more than one minimal spanning set), a basis is a less-sized spanning set, to mention just a few. For the third question, one participant wrote conversely that "If S is independent, and $S \cup \{\alpha\}$ is also independent, then α comes out of span S". Some responses to the third question were unanticipated. In the Fill-in-the-Blank (d) of question 3, one wrote that if S has no redundant thing, and α comes from S, then $S \cup \{\alpha\} = S$ has no redundant thing. Some even wrote, α does not exist, so that $S \cup \{\alpha\} = S \cup \varnothing = S$ has no redundant thing. While discussing, a student said, {tea leaves, sugar, milk} may be regarded as a basis for making *chai*." Further, it turns out students discussed the proportions/ mixture of each of them to make *chai* of particular taste. If the proportion is changed even a bit, the taste changes. Essentially, an important result that, "Each vector can be written *uniquely* as a linear combination of basis vectors" was being discussed. During the discussion, it was easy to explain in terms of redundant vectors why the empty set $\varnothing$ is an LI set.

A metaphor must be chosen very carefully for a concept. A "wrong" metaphor may be misleading. It may also happen that a metaphor is useful for one person, but not for some other person, as recorded here. For the third question, when asked to darken the region with vectors that can be made with the vectors of S, many respondents darked only (span S) \ S. When asked the common reply was that the vectors of S are already there, we need not make them. The metaphor may have misled them not to consider $S \subseteq$ span S. One responder wrote that if S has no redundant thing, and $\{\alpha\}$ is linearly independent, then $S \cup \{\alpha\}$ is linearly independent. When asked he said that the union of two linearly independent sets is linearly independent, because none of the two sets have redundant things, so their union does not have any redundant things. However one of his fellow friends fixed the bug, saying "Suppose one person has a glass of milk and a packet of sugar, and another person has a packet of tea leaves and a packet of sugar. Both want to make *chai*. They can make it only collectively, and then one packet of sugar would be redundant." One responder replied for the first question that if a smaller set has a redundant thing, then that thing may be essential for the bigger set. This reply is exactly opposite to the intended reply. When I asked its elaboration, she said, "Suppose we bought a new TV set for our house, because we are bored of watching the old TV set. The old TV set is redundant in our house. It may happen, someone in our colony does not have any TV set, and if we give the old TV set to that person, it won't be redundant for him/her. Thus a redundant thing in our house need not be redundant in our colony." During the discussion, a student said, "In set $\{a, b, c, d\}$, if a is redundant, and $\{b, c, d\}$ is LI, then a is the only redundant element, so *only* subsets of $\{b, c, d\}$ are the LI subsets of $\{a, b, c, d\}$." This need not be true! The last two alternative conceptions are dealt with again in the next section for the last stage of the FAR guide.

CONCLUSION

The findings reveal that metaphors may act as a pertinent tool for making classroom learning more meaningful, and contextual, so the students are likely to reinforce their thinking capacities and construct meaning of the concepts based on their own experiences. The findings also reveal that the use of metaphors has another side also; there are advantages, but also "disadvantages."

We encountered students' various alternative conceptions in the last paragraph of the Findings section. The last two of them are worthy to discuss, as an illustration of the last stage of Reflection. For the second last alternative conception in the previous section, we can search for some other metaphor, like saying a blend-of-other-vectors-in-the-set, instead of redundant vector, which probably helps in overcoming the problem. For the last alternative conception, a traditional list of most basic human needs for life is food, shelter and clothing. Suppose there is a person who has all of them. Assume his/her shelter is a small hut. If someone offers him/her a big house, then which of the two shelters is redundant? Small hut or big villa?! It depends on the person's desires. We are more interested in knowing that if one thing is redundant than which one is redundant. We are concerned with the number of redundant things. Further, the number of redundant things in an SS can easily be related to the dimension of vector spaces. As discussed earlier, due to social discomfort, we may avoid using the term "rich-poor", or if used, our intention should be clear among the students that these are used just for a quantitative purpose only.

"Multiple analogies are better with each analogy selected for the concept it explains best" (Harrison & Treagust, 2006). As an illustration, we can use multiple analogies for LI sets; one with view of redundant vectors, and one with of blend-of-other-vectors-in-the-set. The former helps in explaining question 5 of the questionnaire, while latter may be more helpful for question 1. Caution should be exercised because of the limitations that metaphors pose when over generalized. "Of course analogies have to be used very carefully, thoughtfully and always, always, always as a side dish to the main course of mathematical reasoning" (Sarina & Namukasa, 2010).

To wrap it up, the article talks about the research questions raised in the introductory section in relation to linear algebra through various stages of the FAR guide, and it further envisions an extension of the metaphor-linking based research of linear algebra, research that includes determining various metaphors and their limitations through empirically collected data for more advanced topics (Sweeney, 2012).

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