EXPLORING THE USE OF DEDUCTIVE LOGIC IN GEOMETRY AS A TOOL FOR COGNITIVE GROWTH

Preety N. Tripathi
State University of New York at Oswego, Oswego, NY, USA
preety.tripathi@oswego.edu

This article presents some initial results from a study focused on promoting students' cognitive growth by building their deductive thinking skills in a college geometry course. The strategies used were ruler and geometric constructions and deductive proof.

INTRODUCTION

Historically, it is widely believed that learning mathematics fosters learners' cognitive development, and their problem-solving skills. Globally, mathematics is embedded within the school curriculum largely because mathematics is perceived as a subject that can help humans to learn to reason deductively, and apply such reasoning skills to everyday life. How to foster these reasoning skills? Most educators believe that deductive reasoning is a skill that must be acquired through careful and persistent nurturing. For students specializing in mathematics, by the college level, curricula include axiomatic reasoning and logic structures that are the foundations of mathematical proof, indicating that fostering cognitive growth by honing deductive skills is an important goal for mathematics education.

In this article, I present my work with a group of undergraduate students in a college geometry class. In this course, my goal was to intentionally foster students' deductive thinking skills by including tasks that would specifically target those skills. Thus, I intended to a) discuss the axiomatic development of geometric ideas so that students would appreciate the underlying structure, and use the structure to establish proof, b) include straightedge and compass constructions on a regular basis – students would do the constructions and prove that their constructions were correct, and c) encourage students to come up with their own proofs, with the intention of developing their deductive reasoning skills.

Specifically, the questions for my study were: (1) How can students be helped to appreciate and utilize the axiomatic development of Euclidean geometry? And (2) How can students' abilities in deductive logic be enhanced through construction and proving activities in geometry?

THEORETICAL FRAMEWORK

The work presented in this article is based on a framework supported by research in two broad areas of mathematics education: a) research on cognitive development in mathematics in general, and on the cognitive development of proof, in particular, and b) research on geometric thinking. Specifically, I was interested in helping students to reason deductively and to understand and appreciate the basic underlying axiomatic structure of geometric thinking. In doing so, I hoped that students would gain cognitive skills in mathematics and more specifically, in geometry.

One of the basic premises of my work is that the two seemingly disjoint approaches in mathematics – empirical intuition and deductive logic – are two pillars on which cognitive development in mathematics rests, and it is the interplay between these two aspects that engenders learning. This viewpoint has led to research on how best to support the dialectic between these two areas of distinct yet complementary individual experiences. Students and professional mathematicians, each in their own way, rely on the constant interaction of both these strategies in order to advance mathematically. Lakatos (1977), an early proponent of this viewpoint, asserted that mathematical development is a result of the constant interplay of empirical observation and formal mathematics, even coining the term “abduction” to describe the synthesis. Clarifying this perspective, Schoenfeld (1986) pointed out that empirical knowledge and deductive knowledge are mutually reinforcing, and each enhances the other in significant ways. In particular, I believe that geometry offers an ideal vehicle in which to transact and observe the cognitive growth stimulated by the constant dialectic of empirical reasoning and deductive logic. There are several reasons for my belief.

First, in general, geometrical entities are easier to visualize and manipulate for students than other mathematical objects, and the access to visual representations facilitates the development of reasoning skills based on properties and attributes of geometrical concepts. Secondly, Euclid’s approach to geometry, outlined in the *Elements*, was constructive. Many of Euclid’s theorems are exercises in construction, based on deductive logic. Further, the constructions are accessible for students, most of whom are introduced to the basic constructions in school geometry. As students develop proofs for these results, they experience the element of problem solving. Thirdly, curricula in geometry have been quicker to evolve towards facilitating the interplay between the areas of empirical experience and deductive reasoning. Wirszup (1976) declared that (deductive) proof cannot be meaningful until the entities manipulated in the proof are meaningful. Thus, early on, educators broadly recognized that purely axiomatic initiation into the study of geometry is bound to be largely unsuccessful, and found ways to incorporate hands on experience with geometrical objects at the earlier stages of schooling via tangible objects, visual representations, and real-life experience. I believed that straightedge and compass constructions offer an excellent opportunity to combine these experiences and engage students.

The ground breaking work by Dina van Hiele-Geldof (1957) and Pierre van Hiele (1957) provided further support for a pedagogical approach to geometry that relies on enhancing students’ experiences with geometrical entities, and reflecting on properties of the entities in an interactive manner. The van Hieles provided empirically based description of five stages of geometric learning that delineate the stages or levels that learners go through when developing ideas related to geometry. Modified from Lee, (2015), the levels may be articulated as follows:

- *Level 1*: Visualization: Students can recognize and classify shapes based on visual characteristics of the shape. They are unable to articulate properties of shapes.

- *Level 2*: Analysis: Students can identify some properties of shapes, and use appropriate vocabularies. They cannot use the properties for logical deduction.
- *Level 3*: Informal deduction: Students know the relationships among properties of geometric objects and are able to do informal logical reasoning. They cannot create formal proof.
- *Level 4*: Deduction: Students know the deductive systems of properties and can create formal proof.
- *Level 5*: Rigor: Students can do analysis of deductive systems and compare different axiom systems.

The van Hiele levels are hierarchical but non-discrete, and each subsequent level draws upon understanding built at the previous level. Students must gain sufficient experiences at one level before proceeding to the next, and students will function at different levels simultaneously, depending on the concept. In laying out these principles, the van Hiele model of geometric thinking is useful in two significant ways: 1) The model is useful in understanding where students are situated in terms of cognitive development, and has been used by researchers for this purpose, for example, by Mayberry (1983), 2) the model may be used to develop a pedagogical approach that helps students to transition from an earlier to a later stage, thus moving towards cognitive growth. In their study describing the van Hiele levels of geometric reasoning among students, Burger and Shaughnessy (1986) verify that the levels were useful in describing students' thinking processes on geometric tasks, and that the levels could be characterized operationally by student behavior. Their work suggests that the levels are useful in making pedagogical decisions about students' development in geometry, and designing tasks that aim to raising students to the next level.

The findings of the above research propelled me to use the van Hiele model to help me ascertain where my students were in terms of geometric thinking, and designing experiences that would help them to advance their geometric thinking to the next level. In their report, Tall et al (2012) traced the long-term cognitive development of mathematical proof. Their framework is initiated from perception and action, and evolves through proof by embodied actions and classifications, geometric proof and operational proof in arithmetic and algebra, to the formal set-theoretic definition and formal deduction. The research provided me with some pointers on how to design tasks that specifically foster cognitive growth.

DETAILS OF THE STUDY

For my study, students were drawn from a college geometry class that I taught in the mathematics department of a state university in central New York. Many of the students were preservice secondary school teachers for whom the course was required; almost all of these students had mathematics as a second major. The other students in the class were mathematics majors who were taking the course as an elective. I encouraged the students to work in pairs or in small groups while working on construction problems or proofs. I collected data by observing student work, by writing detailed notes after class, and by taking pictures of student work.

STRUCTURE OF THE COURSE

Most students in the class had taken a geometry course at the high school level. Based on their initial work in class, and on a homework assignment, I found that a majority of them were (mostly) at level 2 of the van Hiele model. They had understanding of properties and attributes of shapes, and tried to reason based on the properties. However, they reverted to the visual aspects of the shape when they encountered an obstruction in the problem-solving process, indicating a reversal to level 1 of the van Hiele model. A few of the students – about 3 or 4 – could be considered to be more at level 2. They tried to enunciate abstract definitions. Sometimes, they offered logical implications, based on the definitions. 1 out of the 22 students functioned mostly at level 3 of the van Hiele model. He was able to use deductive logic to create informal proof but often needed help with formal proof. He tried to incorporate new theorems into an existing network of geometric knowledge.

My objective in the course was to find ways to help students transition to level 3 of the model. This is the level where students think deductively, and I believe that if students became proficient at this level, then they would be sufficiently prepared for the courses that were to follow. To promote students' deductive skills, I chose largely two strategies (1) proof-based activities involving increasingly complex results that would include deductive logic, and (2) ruler-and-compass based constructions, that would include developing the construction, and justifying it. Both these activities would engage students in using the basic axioms and results of geometry, and would evolve as they added more results to their repertoire. Thus, I designed activities on an everyday basis that were geared towards these two strategies.

In the first few weeks of the semester, we laid out the basic foundations of Euclidean geometry contained in the first book of the *Elements* – beginning with the 23 definitions, the five common notions and five postulates, and the early propositions (http://aleph0.clarku.edu/~djoyce/java/elements/bookI/bookI.html#cns). Among Euclid's first 12 propositions, seven involve construction. The students did these constructions, and for each one, students would prove the constructions were correct by using previous propositions and the common notions and postulates. A crucial component of the class was the small groups in which students worked – they pushed each other to justify their reasoning by asking questions such as "which result are you using (for a given step)?"

In addition, the textbook we were using (Libeskind, 2008) also facilitated the process. As in Euclid's *Elements*, the book formulated constructions as theorems requiring proof. For example, theorem 1.12 in chapter 1 characterizes an angle bisector as follows: A point is on the angle bisector of an angle if and only if it is equidistant from the sides of the angle. This theorem came up early in the course, and followed a section in which students defined and studied the properties of kites. Once students understood such a statement, they began to see that the statement reflected their construction, and was a tool to help them analyze their steps. Thus, students knew (from theorem 1.12) that in order to construct an angle bisector, they needed to construct the set of points that was equidistant from both arms of the angle. Having studied the properties of kites, some students knew that the diagonals of a kite are angle bisectors for the angles that they connect. They used this idea to construct a kite using (parts of) the two arms of the given angle (see figure 1).

Here, students used the result that the diagonal (BD) of a kite (DEBF) bisects the angles at the vertices that it connects. Thus, students drew a circle with center B, and radius BE. This construction created the sides

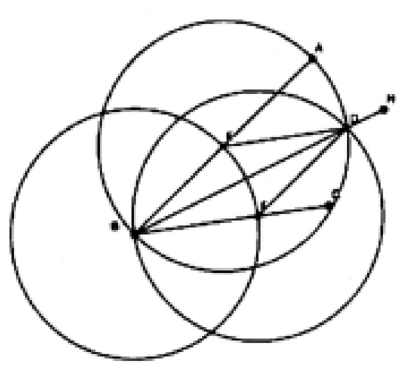

Figure 1

BD and BF of a kite. Students then completed the kite by drawing circles centered at D and F respectively with the same radius DE = FE (actually, students more often drew a rhombus). Thus, EB is the angle bisector of angle ABC.

Further, the exercises in the book required students to construct various geometric entities, and then prove that the construction was correct. The first few weeks helped to set the tone of the class, and to master the basic constructions.

Another important strategy that the book suggested for solving construction problems was outlined in three steps: Step 1) Assume that the construction is done 2) Analyze the construction and the shapes in it for various attributes and properties. Step 3) Use the properties and attributes to work backwards and carry out the construction. I strongly urged the students to use this strategy. An early attempt by students to utilize this strategy is described below.

Episode 1

Question from the exercise in Libeskind (2008, p. 38): A circle such that each side of a triangle is a tangent to it is called an inscribed circle. Assume that a tangent to a circle is perpendicular to the radius at the point of contact. Explain how you will find the inscribed circle, and then construct it. (Question restated in Figure 2)

Given: Δ MNO.
To construct Inscribed circle in Δ MNO

Figure 2

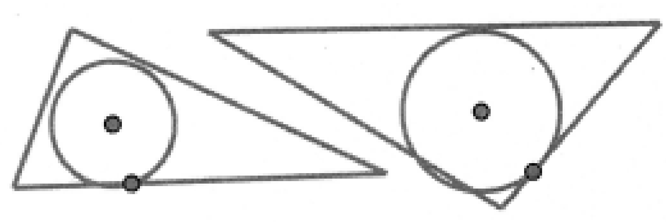

Figure 3

Students attempted this problem in pairs or in groups of three. Some students immediately drew a triangle and then tried to construct a circle inside it by guessing a center and a radius. Figure 3 shows the typical attempts. Clearly, this strategy did not work. I reminded them of the 3-step strategy outlined above. For step 1 of the process, students needed to assume that the shape had been constructed, and draw what the resulting construction would look like.

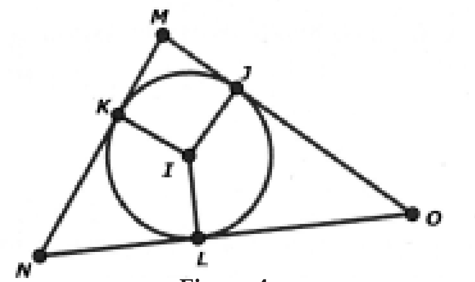

Figure 4

Soon, students realized that it was easier to draw the circle before they drew the triangle. One pair of students, Alexa and Joey, (pseudonyms), who were working together, drew the diagram in Figure 4 as a rough sketch of what the picture would look like if it were actually constructed. Thus, they first drew the circle (I), then chose three points (J, K, and L), then drew the three tangent lines. Then they proceeded to analyze the picture.

Their conversation was as follows:

Alexa: So, how do you think these (pointing to IJ, IK, IL) are related? I mean, I know they are radii but what else?

Joey: Yeah, there has to be something else, right? (They looked to me, and I stayed quiet)

Joey: Looks to me like these are perpendicular bisectors or something …

Alexa: You mean of the sides? Hmmm … yeah, but like, NL looks much shorter than OL. (After some thinking) So, did we draw this wrong? (Long pause)

Joey: But the sides have to be tangents, and they are, aren't they? (This last question was addressed to me, and I stayed quiet)

Joey: (after a pause) Well, the tangents have to touch at one point.

Alexa: (thinking) But that's not how we defined it. Look (turns back a few pages to show the highlighted definition in her notes). It says: the tangent is a line that is perpendicular to the radius at the point of contact. So, we have to assume that these lines are perpendicular to the radius.

After a while when I came back to them, they had marked the right angles IKN, ILO, and IJO.

Discussion: Analyzing this episode, I believed that Joey's purely visual reasoning indicated level 2 thinking – in the picture, he thought that the radii looked like bisectors (though even this was contradicted by Alexa). Similarly, their initial conception of a tangent being a line that "touches" a circle ("touching" being an undefined concept) was indicative of level 2 thinking. However, Alexa's reference to the definition as a step towards their construction indicated a readiness to move to deductive reasoning. I believe that reasoning on the basis of the definition was an important step in advancing students' geometric thinking, and showed a move towards level 3.

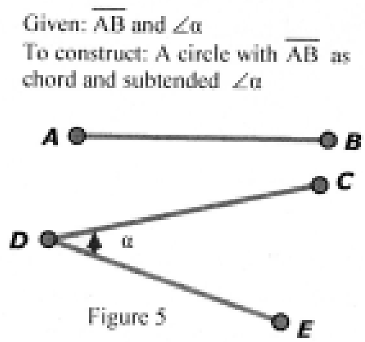

Figure 5

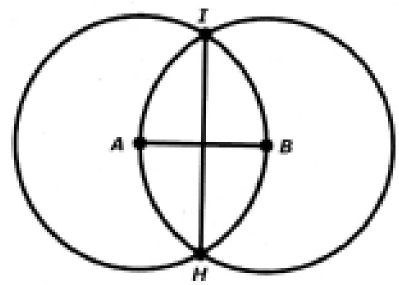

Figure 6

Episode 2: A few weeks later, students worked on the following construction problem:
Given an angle*a*, and a line segment *AB*, construct a circle that has *AB* as a chord with *a* being the angle subtended by *AB*. I asked students to draw their own angle *a*, and a chord *AB*.

When students worked on this problem (restated in Figure 5), they quickly pointed out that based on postulate 3, they needed the center and the radius of the circle in order to construct it. After some time, I joined a pair of students, Stephanie and Dana (pseudonyms), to observe their work.

Stephanie: Should we begin with the angle, or the chord?

Dana: So, if we could draw the angle, like this (she drew the vertex of *a* at the top of the page, and the arms extended downwards), then I can try to fit the chord *AB* (indicating line segments between the arms)? Right?

Stephanie: So you mean by *measuring*? (Her emphasis) Ha, ha! Not allowed! (They both looked at me, and laughed) Against the rules, right?

I: You've got that right! (We laughed together).

Dana: (thinking) OK. One thing I know. The center lies on the perpendicular bisector of a chord. We've proved and used that before – I remember that.

Stephanie: I agree. So, let's draw that. (When I came back a few minutes later, they had constructed a copy of AB, and constructed its perpendicular bisector IH – see figure 6). Now, how do we find the centre (of the circle) on this line (pointing on IH).

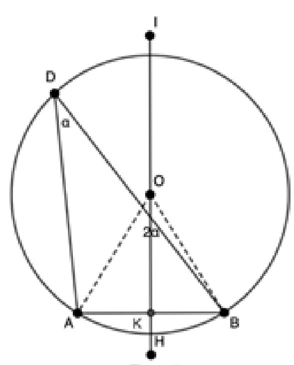

Figure 7

At this stage, they tried several unproductive strategies. Then, they overheard another student, Ben (pseudonym) talking about assuming that the figure had already been constructed. So, Stephanie and Dana decided to try that strategy, and drew the picture as in figure 7 (without the dotted lines). Then, they flipped the pages of their notebooks, considering various other results about circles that we had proven in class. Finally, they found the theorem that stated that for a given subtended angle on the circumference of the circle, the central angle subtended by the same chord is twice the subtended angle on the circumference (that we had done in a previous class).

Stephanie:	(pointing to the diagram for the theorem) Doesn't this look like the picture we are trying to construct?
Dana:	Yeah, that's what I was thinking. We can do this (She drew the dotted segments OA, OB)
Stephanie:	Looks like we have two congruent triangles here.
Dana:	Where?
Stephanie:	(labelling K) See, right here. Triangle AKO, and triangle BKO. Because radii (OA, OB) are equal, AK = BK, and angles at K are right angles (pointing to each object).
Dana:	So, hypotenuse-leg (writing HL). OK, I see. So now angles at O (angles AOK, BOK) are equal to a. Hm, that is nice, isn't it? Because if we now make AD parallel to OH …
Stephanie	(interrupting) But AD is not parallel to OK.
Dana:	Yeah, but we can choose it that way because all these angles subtended on AB are equal, so we can pick one whose arm is parallel to OK.

The students completed their construction using this idea.

Discussion: This was a challenging construction problem for students; they needed to use two theorems (that were relatively new to them) in conjunction. In order to combine these results they had to use deductive reasoning. The students would need to recognize and use the relationships between (a) the centre of a circle, and the perpendicular bisector of a chord and (b) between the angles subtended by the same chord at the centre and on the circumference. Students knew the statements of the related theorems but utilizing the statements to construct the centre of the circle forced them to apply the theorems in a new context Dana and Stephanie were able to use the relationship in (a) quite smoothly indicating level 3 thinking; however, the relationship in (b) above was a little more subtle. The crucial steps in recognizing and using that relationship came when they overheard Ben (thus being prompted by an outside source) and seeing the picture for the theorem. So, they were able to reason through the second part of the problem – a progression towards level 3 thinking; they even did some formal proving activities unprompted (recognizing congruent triangles and identifying the criterion) which is classified as level 4 thinking in the literature (Lee, 2015; Fuys, Geddes, and Tischler (1988)).

CONCLUSION

At the beginning of the class, students presented their arguments citing properties of geometrical objects that they were very familiar with such as triangles, rectangles or parallelograms. We then discussed the initial part of the *Elements* in some detail, including some of the definitions, and why definitions were useful; in particular we discussed how the foundations of geometry as laid out by Euclid, were instrumental in moving learners forward in thinking deductively.

As evident from the two episodes described in this paper, there were marked differences in the ways that students built their constructions at the beginning and then towards the end of the semester. The students were definitely connecting their ideas more by using propositions and results that we had proved in class, indicating that they were ready to progress to a higher level of the van Hiele model. In the context of the model itself, it was evident that even for the same problem, (the same) students could function at more than one level simultaneously. As their instructor, this was exhilarating; as a researcher, this made the work of classifying the students by levels a little more thought provoking. The inter-student interaction was very helpful in maintaining a productive atmosphere in the class where students discussed ideas and defended their arguments with each other. Certainly, it gave me hope to extend my work and my efforts towards more work in this area.

REFERENCES

Burger, W.F. & Shaughnessy, J.M. (1986). Characterizing the van Hiele levels of development in geometry. *Journal for Research in Mathematics Education*, 17(1), 31-48.

Fuys, D, Geddes, D., and Tischler, R. (1988). The van Hiele model of thinking in geometry amon adolescents. *Journal for Research in Mathematics Education*, *Monograph*, 3, 1-196.

Joyce,D.E.(1996). Euclid's Elements: Book 1. Retrieved from: (http://aleph0.clarku.edu/~djoyce/java/elements/bookI/bookI.html#cns)

Lakatos, I. (1977). Proofs and refutations (revised edition). Cambridge, Eng.: Cambridge University Press.

Lee, M.Y. (2015). The relationship between pre-service teachers' geometric reasoning and their van Hiele levels in a Geometr's sketchpad environment. *Journal of KSME Series D: Research in Mathematics Education*, 19(4), 229-245.

Libeskind, S. (2008). Euclidean and transformational geometry: A deductive inquiry. Jones and Bartlett Publishers International, Toronto, Canada.

Mayberry, J. (1983). The van Hiele levels of geometric thought in undergraduate preservice teachers. *Journal for Research in Mathematics Education*, 14(1), 58-69.

Schoenfeld, A.H. (1986). On having and using geometric knowledge. In J. Hiebert (ed.) Conceptual and Procedural Knowledge: The Case of Mathematics (pp. 225-264). Lawrence Erlbaum Associates, Hillsdale, NJ.

Tall, D., Yevdokimov, O., Koichu, B., Whiteley, W., Kondratieva, M. & Cheng, Y.H. (2012) Cognitive development of proof. In Hanna, G. & De Villiers, M. D. (Eds.) Proof and Proving in Mathematics Education:

The 19th ICMI Study, pp 13-49. New York, NY: Springer.

Van Hiele-Geldof, D. (1957). *De didaktick van de meetkunde in de eerste klas van het V.H.M.O*. Unpublished doctoral dissertation, University of Utrecht.

Van Hiele, P. M. (1984). English summary. [The problem of insight in connection with school children's insight into the subject matter of geometry.] (Unpublished doctoral dissertation, University of Utrecht, 1957) In D. Fuys, D. Geddes, & R. Tischler (Eds.), English translation of selected writings of Dina van Hiele-Geldof and P. M. van Hiele (pp. 237-241). Brooklyn: Brooklyn College.

Wirszup, I. (1976). Breakthroughs in the psychology of learning and teaching geometry. In J. L. Martin & D.A. Bradbard (Eds.), Space and Geometry: Papers from a research workshop (pp.75-97), Columbus, Ohio: ERIC Centre for Science, Mathematics and Environmental Education.