EXPLORING MATHEMATICAL EXPLORATIONS

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A Mathematical Exploration is a loosely defined problem situation that has the potential to generate multiple questions, at least some of which allow for further questioning that lead to mathematically significant results, through means that are mathematical. For most students, such explorations offer a new and friendly perspective of mathematics and it is important to provide them with such educational opportunities. What does it take to enable and sustain mathematical exploration in a classroom? Based on the experience of taking four different cohorts through an exploration, this paper addresses this question and provides some preliminary answers. The larger study, of which this paper is a preliminary step, hopes to understand the exploratory process better so as to be able to come up with a 'local instruction theory' for explorations.

INTRODUCTION

Skovsmose (2001), in his essay Landscapes of Investigation, differentiates two different learning milieus. According to him, traditional mathematics education falls within what he calls the exercise paradigm, where a mathematics lesson is occupied with two kinds of activities–a teacher presenting some mathematical ideas and techniques and students working on some related exercises. The relative proportion of time occupied by these activities may vary, but the activities themselves don't. He contrasts this with 'Landscapes of Investigation', which invite students to formulate questions and look for explanations, rather than solve the exercises set by the teacher or the textbook and is characterised by classroom practices that support an investigatory approach. "When the students in this way take over the process of exploration and explanation, the landscape of investigation comes to constitute a new learning milieu." (Skovsmose, 2001, pp.125)

In the Indian context as well, we see this dichotomy with the policy documents envisaging the latter paradigm and the former prevailing in classrooms. The National Focus Group (NFG) Position Paper on Teaching of Mathematics talks of the need to ensure learning environments which invite participation, engage children in posing and solving meaningful problems, offering them a sense of success and to "liberate school mathematics from the tyranny of the one right answer found by applying the one algorithm taught", through a multiplicity of approaches, procedures and solutions (NFG, 2006).

Polya (1945) talks about the need to challenge the curiosity of students by setting them problems appropriate for their mathematical knowledge and helping them solve the problem with stimulating questions, thus giving them a "taste for and some means of independent thinking". Without this a student may miss out on the opportunity to know whether he/she has a taste for mathematics at all. We see mathematical explorations as a means to provide such an opportunity.

'Mathematical Exploration' is an open-ended and loosely-defined problem situation, that involves students asking their own questions, choosing the ones that interest them, following different paths to find answers and asking further questions.

The Oxford dictionary defines exploration to be "The action of travelling to or around an uncharted or unknown area for the purposes of discovery and gathering information; the action or activity of going to or around an unfamiliar place in order to learn about it; expedition for the purpose of discovery'' (OED Online, 2019). This is a relevant image for mathematical explorations by students as well. Critical to exploration is the unfamiliarity of the terrain. It might have been charted extensively by others, but the fact that it is new terrain for the student is important. Navigating the terrain for the purposes of discovery and gathering information is both challenging and engaging for the students. It is here that the availability of reliable maps, and a tour guide can make a big difference. It is also clear that the terrain may be difficult but not forbiddingly so, lest the explorer give up too early. Existence of vantage points from which one can take an overall perspective of the route travelled and pathways ahead greatly helps the exploration.

Similarly, an activity that is intended to set forth a mathematical exploration should be unfamiliar enough so as not to have a learnt solution and at the same time relevant, engaging and approachable. The entry point to the activity should be accessible to every student and ideally there should be multiple entry points. At the same time, the activity should have the potential to challenge the more interested students and keep them involved. In other words, the activity should have a 'low threshold, but high ceiling' (LTHC). The activity should have the potential to branch out into multiple trajectories, at least some of which have the potential to raise deeper questions, which the students have the necessary resources to solve. These solutions themselves could generate further questions, some of which may not be as yet answered by the general mathematical community. A Mathematical Exploration provides opportunities to raise questions, the answers to which are hitherto unknown to the explorer and sometimes even to the community at large and find answers by engaging in the processes of mathematics.

In taking over the process of exploration and explanation, students are functioning like 'little mathematicians', posing questions that interest them, and engaging with the processes of the discipline like coming up with conjectures, visualising, representing, estimating, justifying, generalising, and overall experiencing the joy of making their own discoveries. This gives them a different kind of experience of doing mathematics , different from the fearsome and anxiety-inducing one that they are used to (Ramanujam, 2010). In the learning milieu created by explorations, every child has an opportunity to succeed at some level. In the Indian context, one of the first attempts at creating such a learning milieu can be seen in Eklavya's Prashika Project, though at the primary level (Agnihotri, Khanna, & Shukla, 1994).

RELATED RESEARCH AND ENSUING QUESTIONS

Problem posing and solving and engaging in the processes of mathematics or 'thinking mathematically' are three key aspects of a Mathematical Exploration. Understanding what is involved to enable and sustain an exploration in the classroom calls for taking a closer look at each of these three aspects and how they come together in an exploration. In addition, one also needs to understand the role of the students, teachers and the material in the process. We present a brief overview of research on these aspects.

While there has been considerable research on Problem Solving and multiple aspects of it, (Polya, 1945; Törner, Schoenfeld, & Reiss, 2007), problem posing has garnered attention in recent years as well (Brown & Walter, 2005; Singer, Ellerton, & Cai, 2015). In their seminal work on Mathematical Thinking, Mason, Burton, & Stacey, (1982) delineate the practices involved in thinking mathematically and identify specialising and generalising, conjecturing and convincing, imagining and expressing, extending and restricting, classifying and characterising as the core mathematical processes. Others (Bell, 1976; Watson, 2008; Schoenfeld, 1985) have slightly different characteristics of the processes involved.

There has also been a closer look at these disciplinary practices and characterisation of progression in mathematical thinking. Zandieh and Rasmussen (2010) and Rasmussen, Zandieh, King, and Teppo (2005) exemplify how students in undergraduate classrooms engage in disciplinary practices like defining and symbolising. These papers evolve a framework to describe the stages in their progression.

A natural question arises as to whether a similar framework can be created for secondary school students, characterising the progression of thinking and the development of disciplinary practices during explorations. Some spadework in this direction can be seen in Cai and Cifarelli, (2005) and Cifarelli and Cai, (2005). Through a case study of two college students the authors examine how an interplay of sense-making, problem-posing and problem solving sustains an exploration and initiate the development of conceptual frameworks and research tools to capture mathematical exploration processes. They identify two levels of reasoning strategies in the process– hypothesis driven and data driven. However these are not sufficient to develop a 'local instructional theory' (Gravemeijer, 2004) for mathematical explorations that describe potential learning trajectories through which students might progress, thus functioning as road maps or frames of reference for teachers who want to engage their students in an exploratory activity.

We note that the idea of a 'local instructional theory' is closest in spirit to this work. While such a theory has been developed in a specific content area, and for processes such as definition and symbolisation in the content area, our eventual goal is to develop a similar theory for explorations at the secondary school. However, while theorisation is the principal aim of this line of research, the account presented here is too preliminary for any theory-building as yet.

Jaworski's work on the role of the teacher in an Investigatory classroom, (Jaworski, 1994) identifies some generic pointers like sensitivity to students and need for mathematical challenge to sustain an investigation, but does not answer questions like - What moves on the part of the teacher help or hinder an exploration? At what stage in the progress of an exploration is an explicit hint helpful and at what stage is it a better choice to let the students struggle to find their own path? What is the nature of preparation that a teacher should have before starting on an exploration with students? These and related questions on other enabling/hindering factors for an exploration form the backdrop of this study.

STARTING POINT AND POTENTIAL TRAJECTORIES

Combinatorics offer many rich possibilities for explorations (Maher, Powell & Uptegrove, 2011). The 'starting point' being considered for this paper is a puzzle where students are invited to arrange numbers 1-6, using each exactly once, in circles arranged along the sides of an equilateral triangle, in such a way that the sum of numbers along the three sides are equal (Trotter, 1972). That there are four such distinct arrangements provides an access point to all students into the task since they can discover these by mere enumeration and sets the ball rolling with questions such as What does it mean to say distinct solutions?, How many distinct side-sums are possible? What are the upper and lower-bounds for the side-sums?, Are these the only solutions? How does one prove that there are exactly 4 solutions? What patterns can be seen in the solutions? How can one 'transform' one solution into another? How many solutions (non-distinct) can be obtained by rearranging one given solution? How is the side sum related to the corner sum (which is the sum of the three numbers placed at the vertices of the triangle)? Can the side sum ever be twice the corner sum?

While staying with an equilateral triangle formed with 6 circles, one can ask further questions like – what if a different set of numbers are used instead of 1 – 6? Do the numbers have to be consecutive in order for solutions to exist? Will there be 4 distinct solutions, whatever the choice of numbers? If not what condition should the numbers satisfy for the existence of 4 distinct solutions? What conditions should the numbers satisfy for the existence of a solution at all?

The more interested students can engage with questions like – What if the circles are arranged in the form of a square? Or a pentagon or any other polygon for that matter (Trotter, 1974)? What if there are more than 3 circles per side? What if the circles are arranged in the form of an open curve like an S or a Z? Which of the questions asked in the context of the initial triangle are still relevant? Can the solutions/methods of solutions used there be generalised to these arrangements?

Thus we see that the 'task' meets the LTHC criteria, generates multiple questions and divergent paths for exploration. Starting from a simple puzzle also ensures the engagement of all students. It also touches upon some significant mathematical ideas like the existence of upper and lower bounds, proofs of existence or non-existence of solutions, generalisable methods that work across a range of problems etc. Students also get opportunities to engage in the processes of mathematics like observing patterns, coming up with conjectures, looking for examples or counterexamples, justifying, generalising etc. This task was tried out with multiple groups of students and similarities and differences in the way it panned out observed.

THE STUDY GROUPS

The groups differed widely in terms of the prior exposure that they had to mathematics. Some details of the different groups are as follows:
Cohort A: about 10 students from grades 8 and 9 of a corporation school

Cohort B: about 10 students from grades 8 and 9 of a low-fee private school with students from disadvantaged socio-economic backgrounds

Cohorts C and D: about 25 students each who were part of a talent nurture camp in mathematics and from fairly affluent backgrounds

In terms of ‘mathematical background’ cohorts A and B can be considered similar, and C and D are similar as well. Cohorts A and B were drawn from ‘typical’ schools whereas cohorts C and D had been identified as potentially ‘talented’ in mathematics and had been through focused enrichment programs for a few days every year for the past 4 years. While it is interesting and important to study the impact of socio-economic context on mathematical explorations in the classroom, this paper does not take up this difficult comparative task and contents itself with the more basic question of whether explorations take place at all, across these contexts, and how they proceed. All four groups spent about 2- 3 hours on the task. This account is based on reflective notes of sessions and audio-recordings of sessions with cohorts A and B.

THE OBSERVATIONS

The way the exploration progressed with each of these four cohorts was distinctive, even though there were pairs of cohorts with similar mathematical backgrounds. While some questions and conjectures came up uniformly across all groups, the arguments that each group came up with in support of these were different. In this section, we highlight some of these similarities and differences.

Finding distinct solutions phase

The four distinct solutions were found out by all four cohorts, some sooner and others a little later. Different approaches to finding the solutions were seen. Though all of them started off with a trial and error method they evolved differently.

With cohort A, fairly early in the trial and error phase, after three solutions were found out, the question “Will all the numbers come as sum?” was raised by a student. The teacher, noting the potential of this question to go beyond finding solutions phase, revoiced it to the whole class. Possibly guided by this prompt, possibly not, one dominant method of looking for solutions that was seen in this group was to try to find an arrangement with a pre-determined side-sum.

Some students from cohort B engaged in a similar kind of reasoning as well, but here a student realised very early on that moving around the numbers in a given solution in a cyclic order yields another solution. Having found this, the group looked for other transformations that could be done to get more solutions from ones already found out. So for this group transforming existing solutions came to become a standard strategy to look for more solutions even when working with numbers 1-6.

Using transformations of existing solutions to generate newer solutions happened with cohort A as well, a little later, but with much excitement. The student who first thought of it claimed ‘ownership’ to the findings

by calling them ‘N’s theorem I’ and ‘N’s theorem II’ on his own, and the whole group toed the line, adopting the same terminology for the rules! They also reached the conclusion that if they find one solution another one comes free by applying the ‘theorem’ and found out some pairings happening among the solutions.
With cohorts C and D the four distinct solutions came up within the first few minutes. Though the solutions came up from different individuals, the four solutions were recorded on the blackboard very soon. The patterns among the solutions and using them to find further solutions did not emerge as points of discussion. However, these groups also had some systematic way in which they generated solutions. Cohort C for example came up with the strategy of fixing the corner numbers, and putting in the minimum of the remaining three numbers in the centre of that side where there is a maximum sum of the two numbers at the corners, and so on.

Thus, examining the different ways in which the finding four distinct solutions to the triangle puzzle panned out with four different cohorts, we notice the following:

- All 4 groups moved from looking for solutions through trial and error to better mathematical ways, though along different paths and at different rates.
- The kind of prompts given by the teacher, or the kind of student findings/conjectures chosen to highlight or revoice to the whole group may have influenced the path the exploration took.
- The sense of joy the students had in finding out something for themselves, the sense of ownership they had for these findings and how they built on these findings was evident in all four groups.

Finding the upper and lower bounds for the side-sum

The attempt to find ‘yet another solution’ to the puzzle soon led all the groups to the conclusion that some side-sums are possible and some are not. Soon all four groups came to the conclusion that side-sums below 9 above 12 are not possible. But they had different ways of arguing this.

For cohort A the initiation to think along these lines came from the question – “will all numbers come as sums?” The first response was that numbers 1-6 have to be ruled out. Soon numbers 7 and 8 also got added to the ‘not-possible’ lists. One of the first ‘arguments’ that came up went something like ‘3, 4 and 1 add up to 8. Now we have to have 6 somewhere and there the sum will go up. We can have a sum of 8 only on one side.” which eventually M refined to “In some circle we have to have 6. Smallest number is 1. On a side there are two more circles. To get 8 there we need to add two 1s (to the 6) and we can’t do that.” However, they did not come up with a similar argument as to why a side-sum of 13 was not possible, in spite of the teacher prompting them to follow a similar reasoning.

Cohort B also followed a similar path, trying out specific combinations of numbers and realising that they would not work and then coming up with an argument for 8 being the lower bound. Like cohort A they did not extend it to the upper bound.

Cohort C and D on the other hand did not give the specific case based arguments, but straight away gave the argument similar to that which M of cohort A had come up with to establish side-sums of 8 or less are not possible and extended it to argue that side-sums of 13 or more are not possible as well. Choosing that side of the triangle where 1 occurs, the maximum possible sum is got when the largest of the six available numbers namely 5 and 6, are on the same side, giving a sum of 12.

Later on in the course of the exploration, student N from cohort A, mentioned in an earlier paragraph, figured out a way of finding out the max sum and the min sum for any given set of numbers. He found out that the max sum was obtained when the larger three numbers occupied the corners and the min-sum when the smaller three of the six numbers were at the corners. He also saw that the smallest of the remaining three should be put in the centre of that side where the larger two numbers are at the corners and vice-versa to equalise the side-sums. Though N didn't explicitly make the connection, this results in expressions for the min sum and max sum given any set of six consecutive numbers.

Here we see that, through exploring possible and impossible side-sums, all 4 cohorts were coming to the important mathematical idea of looking for lower and upper bounds for the side-sum. But the kind of arguments that they come up with show different levels of mathematical thinking. We have students –

- trying out specific cases and concluding from them that certain sums are not possible (without a proper justification),
- coming up with a justification for the impossibility of specific side sums (say 8 and 13) and in principle, extending the same argument for smaller and greater numbers as well,
- coming up with a general expression (well, almost!) for the max and min sums in terms of the given six numbers.

Proving that only 4 solutions exist

This could have been done on multiple ways –

1. by exhaustively considering possibilities,
2. by using parity arguments,
3. having proved that there are only 4 possible side sums, by proving that each of these corresponds to a unique solution,
4. using algebra to establish a correspondence between possible side-sums and corner sums and use this to limit possibilities.

None of the 4 cohorts started on any of these on their own accord. Knowing that method 4 above is extendable to the variations of the problem discussed in an earlier section, the teacher explicitly cued this method, by suggesting that they 'let a,b,c,d,e,f be the numbers 1 to 6 in some order and S be the side-sum' and asking them to write an expression for the sum of numbers on each side in terms of S and a,b,c,d,e,f. With varying degrees of teacher support, cohort B, C and D came up with the relation that

$$3S=\sum_{1}^{6} n + C,$$

where C is the corner sum. All three cohorts inferred that the corner sum has to be a multiple of 3 and used this to limit the possibilities to arrive at a proof. Cohorts B and C explored the generalisability of this method to other alignments of circles as well. This exercise could not be taken to completion with cohort A for want of time.

Here we see the crucial role of the teacher in keeping the exploration going. There are times when the teacher has to make an explicit suggestion instead of waiting passively for students to come up with those critical insights. In this case, with cohort A, the teacher waited unduly long, giving indirect hints to students which they did not catch on. Consequently there was loss of time and the exploration did not progress as much as it did with the other cohorts. The sense of not being able to progress resulted in feelings of frustration in students and their enthusiasm to engage with the exploration waned. Having learnt from the frustration of students and from discussion with others, the teacher changed strategy and made some explicit suggestions in subsequent trials of the module. Thus, it is a hard task for the teacher to balance between giving just enough support so that students stay motivated and at the same time leave sufficient opportunities for them to struggle and discover things for themselves.

CONCLUDING REMARKS

Each of the subsections of the previous section describe three 'instances' in the span of an exploration, each drawing attention to a different aspect that needs closer scrutiny. In the first subsection we describe the movement of all 4 cohorts from trial and error methods to more mathematical ways of solving. This raises the question: what cues (from peers, teachers or materials including tasks themselves) support or hinder this move? The description in the second subsection highlights the different levels of mathematical thinking seen, calling for a clearer description and characterisation of these levels and possible learning trajectories through these levels leading to 'a local instructional theory' of mathematical explorations. The third subsection highlights an instance where a teacher move or lack of it hinders an exploration calling attention to the scaffolding needed and to the timing of provided support.

Even with these limited instances, we can see some essential characteristics of explorations: realisation that there are many solutions leads to the 'how many' questions, and then to the 'how does one know if there are no more' question. Extensions and generalizations occur naturally. The difficulty is also clear: when and why does an exploration stop, or run out of steam? What learning can one carry from one exploration to another? These seem to merit a deeper investigation, and will hopefully contribute to the theorization we seek.

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