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CONCEPT IMAGES OF QUADRILATERALS: A COMPARATIVE STUDY OF VIII AND IX GRADE SCHOOL STUDENTS

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The study attempts to explore the concept images of students regarding quadrilaterals. The analysis is based upon the theoretical frameworks of Fichebein theory of figural concepts, mathematical relationship and classification of quadrilaterals. Findings of the study suggest that in spite of teaching quadrilaterals with a single definition; students came up with variety of definitions which were personal rather than formal. Students used non- critical attributes in non- formal or incorrect definitions. Trapezium and kite received maximum variations in its definitions and images. Findings of the study suggest that hierarchical classifications of quadrilaterals demand attention in school curriculum.

INTRODUCTION

'Quadrilateral' has been described in many ways in mathematics. It is used synonymously for 'Quadrangle' and comes from the Latin *quadric, a combining* form for 'four' and *latus* means sides. 'Quadrilateral' *is defined as 'consisting of four lines, no three of which are concurrent and the six points they determine' (Usiskin and Griffin, 2008).* Due to the etymology of 'Quadrilateral', it has also been considered to be a polygon with four side as found in many textbooks. Quadrilaterals are formally introduced with its naming at elementary level, first in class VI under the heading of 'basic geometrical shapes' and then in class VIII (named as 'understanding quadrilaterals') and class IX (named as 'Quadrilaterals'). The level of sophistication increases from class VI to IX i.e. from empirical to more axiomatic in nature.

There are number of studies specific to quadrilaterals which built upon Van Hiele levels and in relation to other cognitive aspects. Nakahara (1995) found that an understanding of basic quadrilaterals develops accordance with Van Hiele levels, but it is specific to the geometric figure involved, i.e. level assigned to students may be different for different geometrical figures. Further Transition from level 2 to level 3 has been considered slow and problematic by a number of studies because of the distinguishing feature of level 2 i.e. class inclusion- 'interrelationship between two sets when all the members of the first are members of the second e.g. squares is a subset of rectangle' (Owens & Outhred, 2006) whereas identification of features of a shape and grouping figures based upon a single property are important aspects of level 2 (Clements and Battista,1991; De Villier, 1998;Currie and Peg, 1998; Fujita and Jones,2007). Matsuo (1993) suggested that students' consideration of a square as rectangle depends upon the property they are focused on. If their concept definition consisted of a rectangle having four right angles or parallel sides, there is feasibility that students will consider square as a rectangle and if they consider rectangle with two longer and two shorter

sides, square is excluded**.** Researchers aspired to help students overcome difficulties in inclusion tasks, where there is a potential gap between students' geometrical thinking level (Van Hiele, 1986) and task level. For example, the classification of a rectangle as a parallelogram requires Van Hiele's level 3 geometrical thinking (Order), hence students who fail to classify it correctly do not seem to have reached this level (Gal & Lew, 2008). Studies have highlighted the importance of defining and hierarchical classification of quadrilaterals (e.g. De Villier, 1998, 2008; Fujita and Jones, 2007) and argued that formal definitions can be developed at level 3 as at this level, it is expected from students to see the interrelationship between shapes. He has also discussed in his studies that students should be allowed to form visual, uneconomical and economical definitions and should be actively involved in formulations and evaluating definitions. Other studies (Shir & Zaslavsky, 2002; Heinze, 2002; Saenz-Ludlow & Athanasopoulou, 2007; Villier, Govender & Patterson, 2009; Blair and Canada, 2009; Driscoll et.al., 2009) focused on active engagement of students in defining processes such as to analyse definitions; create and critique their own definitions; reason with relationships. These studies have shown that students who are encouraged to participate in defining are able to change their opinions because of interactions as well as justifying and arguing with their colleagues and it leads them to positive gain in understanding nature of definitions and also to think about necessary and sufficient conditions of definitions. Fujita and Jones have emphasized the need of a theoretical framework for the development of definitions and hierarchical classification of quadrilaterals. They argue that a study of hierarchical classification can help in bridging the gap between Van Hiele level 2 and level 3. They also proposed to explore the common cognitive paths of the relationship among quadrilaterals as from their study it was speculated that there could be hierarchical order of the difficulties. For example, they conjectured from their study that it might not be effective to teach the relationship of rectangles and parallelograms before the relationship between rhombuses and parallelograms (Fujita & Jones, 2007).

THE PRESENT STUDY

The objective of this paper is to put forth the concept images of elementary and secondary school students regarding quadrilaterals. This research is in consonance with the assertion that Villiers et.al. (2009) states as 'The classification of any set of concepts implicitly or explicitly involves defining the concepts involved, whereas defining concepts in a certain way automatically involves their classification'. Definition and classification are, therefore, important tools for developing the ability of deductive reasoning and proving. It also plays an important role in identifying new mathematical objects with some precision. However, a number of studies reveal that many learners have difficulties with hierarchical classification of quadrilaterals and related formal defining of such shapes because of cognitive complexities involved in such learning (e.g. Monaghan, 2000; De Villier, 1994).This study considers the complex nature of 'figural concept' as a major factor of learners' difficulties with hierarchical classification of quadrilaterals and related formal defining as also studied by Fujita & Jones, (2007). Fischbein states that 'while a geometrical figure (such as a square) can be described as having intrinsic conceptual properties (in that it is controlled by geometrical theory), it is not solely a concept; it is also an image' (Fischbein, 1993, p. 141) implying that a geometrical figure has characteristics of dual nature in that it is both concept and image and the two are closely interrelated. Learners lack the ability to combine interaction between a concept and its image and hence individuals' *personal figural concepts* are formed that *influences* the classification of quadrilaterals. Personal figural concepts are formed on the basis of and individuals' *personal geometric concept definitions* that are shaped

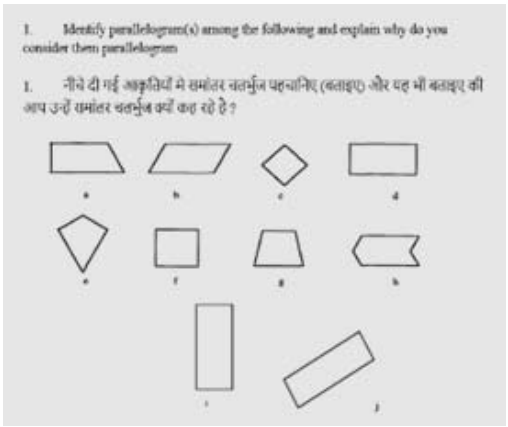
by the *concept images* formed in their minds and which are different from *formal conceptual definition* of a geometrical object.

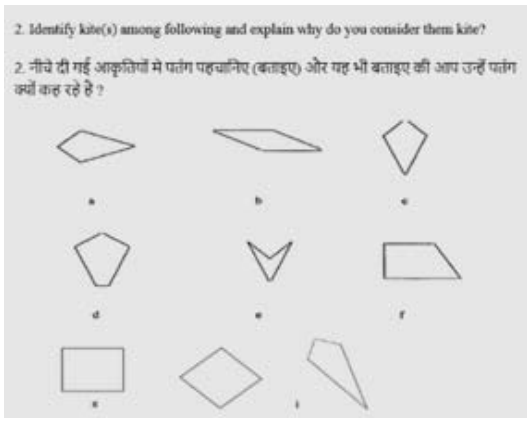
Participants and Setting

The study group consisted of 240 students from 4 sections, each of classes VIII and IX (having 30-35 students in each section) from the two government schools for using worksheets and students' interviews. The government schools had strict admission criteria and had both English, Hindi meduim. However, in spite of having English medium, meduim of instructions commonly used was Hindi. Both English medium and Hindi medium sections were selected for the study. The socio-economic background of students varied from working to lower middle class. Parents' occupations varied from worker in electricity board to, drivers, teachers, farmers, small business to vender etc.

Data Resources

After the topics were taught in the classes by the respective teachers, worksheets related to geometrical tasks were given to students to assess students' understanding of quadrilaterals. The worksheets included items related to (i) Identification of different quadrilaterals and (ii) Defining quadrilaterals.





1. a. Having experiencing from your surroundings and learning from different resources like textbooks, your school or tuition teachers, your friends write down your Ideal definition of the following shapes in your own words:

(i)Polygon, (ii) Quadrilateral, (iii) Parallelogram, (iv) Rectangle, (v) Trapezium (vi) Rhombus, (vii) Square (viii) Kite

b. Also show them diagrammatically

c. Draw and name one example and non-example of these shapes

1. a. आपके आसपास के अनुभव से अपनी पाठ्यपुस्तक, स्कूल अध्यापिका, ट्यूशन अध्यापिका, आपके दोस्त इत्यादि से बातचीत करके या सीख कर, आप नीचे दी गई आकृतियों कि अपने शब्दों में अपनी अभीष्ट परिभाषा दीजिए i) बहुभुज ii) चतुर्भुज iii) समांतर iv) चतुर्भुज v) आयात vi) समलंब vii) समचतुर्भुज vii) वर्ग viii) पतंग

b. इन्हें चित्र द्वारा अंकित जी कीजिए |

c. इन आकृतियों के लिए एक उदाहरण व एक अउदाहरण (जो उदाहरण ना हो) दीजिए |

Figure 1: Worksheets (i) and (ii)

Data Analysis

Data collected through worksheets and interviews was analysed through the process of identifying, coding and categorizing the primary patterns in the data. In the study, students definitions, drawings and reasoning while identifying quadrilaterals were taken into account while analysing students' perceptions. Results were presented in the form of frequencies, students verbatim and their drawings.

Key terms used for the purpose of analysis are as follows:

- Partitional and Hierarchical classification: Partitional classification includes exclusive definitions, which consider concepts involved as disjoint from each other e.g. squares are not considered as rectangle. Hierarchical classification includes inclusive definitions which allow inclusion of more particular concepts as subsets of general concepts e.g. square is classified as a special rectangle
- Formal concept definition: formal is related to mathematically accepted; a concept definition is defined as 'a form of words use to specify that concept' (Tall and Vinner, 1981, p. 152).
- Personal concept definition: It is defined as students' own definitions based upon their own experiences of learning geometry and are different from formal.
- Concept Images: 'The total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes' (Tall and Vinner, 1981, p. 152).
- Formal figural concept: A 'figural concept' which is associated with formal definitions and formal concept images in geometry.
- Personal Figural concept: A 'figure concept' which is associated with personal concept definitions and one's own concept images in geometry.
- Critical attributes are attributes that must be present for the concept to be formed, while non-critical attributes are attributes that may be present but are not required and has strong visual characterstics.

RESULT AND DISCUSSION

In this section, a comparison of concept images of VIII and IX grade students is provided in relation to quadrilaterals. It has been presented in the light of definitions and drawings and reasoning provided by the participants.

Definitions and Drawings of quadrilaterals of VIII and IX grade students

Definitions and drawings of 'basic quadrilaterals' were analysed to explore the students' personal figure concept. Comparison of the images and definitions of 'basic parallelograms' displayed in table 1.1 shows that the majority of students of classes VIII and IX could draw a prototype image (correct image) of quadrilaterals (with an exception of trapezium), however very less students provided their respective correct definitions. Least correct definitions were provided for 'trapezium' and 'kite'. Interestingly, there was not a significant difference in the frequency of respective responses of class VIII and IX.

Class	Class VIII (%) N=120		Class IX (%) N=120	
Figure	**Image**	**Definition**	**Image**	**Definition**
Parallelogram	98	30	99	31
Rectangle	100	33	98	33
Trapezoid	71	26	86	29
Rhombus	100	32	99	34
Square	100	33	100	33
Kite	93	24	98	28

Table 1: Correct responses of drawing and defining of 'basic quadrilaterals'.

When students' definitions of different quadrilaterals were examined, a variety of definitions were observed ranging from listing critical attributes to describing a figure by its name of shape using non- critical attributes. Maximum variations were observed in the definitions of *parallelogram* and *trapezium.* Interestingly, critical attributes were used for defining parallelogram and non- critical attributes were used for trapezium eg. Trapezium as: *A closed figure made up of two figure(s), a triangle and a quadrilateral*, Student H, class 9). Most of the non- formal (natural speech) but satisfying minimal necessary and sufficient conditions were observed for 'kite'. eg. Definition of kite provided by student G of class IX *Upper two lines are equal and lower two lines which meet each other are also equal.* In this definition, he wanted to explain that adjacent sides are equal.

Analysis of definitions also revealed that 84% students of grade VIII and 48% students of grade IX provided definitions having unnecessary or insufficient conditions required for constructing their respective figures. Examples of students' definitions (interpreted) containing unnecessary conditions are (i) *Quadrilateral: It is a four-sided closed figure, it has no properties of sides and angles. (ii) Trapezium: A line which stands on the other line is called trapezium.* An example of definition consisted of insufficient conditions is as follows: *Rectangle: Rectangle is a parallelogram but parallelogram is not a rectangle.*

This definition is insufficient in the sense that, it tells that the rectangle has all the properties of parallelogram but conditions which makes it rectangle was not mentioned in this definition. It was also seen that among rest of the correct definitions provided by students of both grades, most of the definitions contained more than minimal set of necessary and sufficient conditions, termed as *uneconomical* (Villier. et. al. 2009). An example of uneconomical definition shared in natural language by a student of class VIII is: *A figure which has four sides and it is closed from all sides. its opposite sides are parallel. Sum of all angles is 360°.*This definition contains more properties than required to construct a parallelogram.

Some common factors emerged that influences student definitions are categorized as follows:

Language driven

It was interesting to see that 17% of class VIII and 20% of class IX students' definitions of trapezium were influenced by **its name in Hindi language** 'samlamb' (sam means equal, lamb means perpendicular). Another figure in which it was expected to have the influence of language, i.e. parallelogram as stated by Fujita and

Jones (2007) that students may tend to remember about parallel lines from its name 'parallelogram' but they may have limited perception /understanding of it, definitions of parallelograms were not much influenced by its language. Despite of the fact that the name of parallelogram, may reminds students about parallel lines, 22% of class VIII and 25% of class IX students used criteria of 'opposite sides are equal'. Interestingly, criteria of 'parallelness' were used by more number of students of class VIII (23%) than class IX (13%). Many students used linguistic explanation for parallel as equal distance between two lines or lines never bisect (interpreted meaning intersect). In cognitive sense, definition may be influenced by its name but its concept images may be limited. For e.g. rectangle, rhombus and square are termed as special parallelograms under the hierarchical classification of quadrilaterals; limited perception about parallelogram may not consider them as parallelograms.

Non-critical attributes

To define trapezium, 15% of class VIII students and 12% of class IX students used **non-critical attributes** of its prototypical image e.g.

- Trapezium is made up from two triangles and one rectangle
- One triangle and one rectangle
- Look likes rectangle, but sides are not equal
- Looks like a table or pot
- It has parallel sides and perpendicular height.

Some typical parallelogram's definitions of class VIII students (20%) were influenced by its prototypical image like 'opposite sides equal and bends' or 'opposite angles are equal and angles are more than 90°'.20% of class VIII students and 20% of class IX students used non- critical attributes to define kite. Some typical definitions stated by students are:

- *A figure whose all sides are equal, but it is not a square.*
- When we see it, it looks like a prism?
- (Diamond) is called a kite.

Critical attributes and economical definitions

- Most of the definitions of parallelogram were economical definitions using *equal side criteria*.
- Only 18% students of IX class and 7% students of class VIII defined trapezium as a quadrilateral with *one pair of opposite sides parallel.*
- Definitions of rectangle and square had a strong influence of its prototypical image. Therefore, in both the definition, students did not feel to mention about its angles with other critical attributes of their respective images. Angle is an important attribute of rectangle and square as it differentiates parallelogram from rectangle and rhombus from square. Table 2 shows that percentage of students who defined square and rectangles without mentioning about its angles.

Class	Square- all sides equal	Rectangle-opposite sides equal
VIII	38%	43%
IX	25%	15%

Table 2: Students' definitions of square and rectangle

- In case of 'kite', maximum number of economical definitions were written in students' natural speech' i.e. non- formal definitions, for example, non-formal definitions to explain adjacent sides equal by student G of class IX '*Upper two lines are equal and lower two lines which meet each other are also equal'*.

Analysis of students' drawings indicates that most of the students drew prototypical image of these concepts (except trapezium). It was emerged that students used three different ways of connecting an image with its definition:

- Influence of image on its definition
- Influence of definition on its image
- Influence of name of the figure on its definition and hence on image

Influence of image on its definition

The responses of 31% students of class VIII and 17% students of class IX described quadrilaterals without mentioning whether it is a quadrilateral/a four-sided figure/a simple closed curve. The image has such a strong influence that they did not consider it important to mention this while describing its attributes or properties. Student A e.g. gives the following response when asked to define a square *All its sides are equal, opposite sides are parallel, all sides are equal to each other also. All angles are of 90°. Diagonals bisect each other.* The student, however fails to mention that a square is a kind of quadrilateral or a parallelogram or any other shape. Emphasis on '**its'** shows that the student is focused on the image drawn by him/her. In another example of defining a 'kite', Student B responded that *'Kite is shaped like a rhombus when we rotate it"*. In the definition of a rhombus, Student C replied that *all its sides are equal, but its shape is different from a square*. Influence of prototypical image is so strong that the learners feel the need to mention in the definition that it is different from a square. This definition would fall in the category of a partitional classification, i.e. in that it excludes square from the set of rhombus (Villier et.al., 2009). It may be interpreted that concept image of a definition may be responsible for students' exclusive definitions. This connection of image with its definition is has also been borne in the findings of Fujita and Jones (2007).

It was also observed that definitions in natural speech (non -formal) had influence of non-critical attributes of the prototypical images. For example, it was observed in definitions of 33.33 % of class VIII students and 40% of class IX students e.g. Student E defined kite as "*Kite is a shape which has two triangles together and those two triangles are not equal"*. *Another Student F defined kite as* **"***both its vertices are outside, on the top and at the bottom"*. (Here learner wanted to convey that upper and lower vertices of kite are more pointed than left and right vertices). In both the definitions, learners were describing non-critical attributes (attributes which evoke visual image of shape) of kite in its definitions. In the case of the trapezium (42% of class VIII and 30% of class IX) and kite, definitions using non-critical attributes seem to indicate an especially compelling influence on the image. So, for example a student who defines the trapezoid as: *'A closed figure made up of two figure(s), a triangle and a quadrilateral' represents it as in the picture below: (Student H; Class 9)*

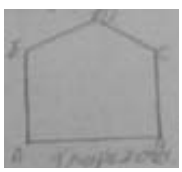


Figure 2: Sample of student's drawing of trapezium

A prototypical image of a trapezium can be decomposed into a triangle and a quadrilateral by joining a vertex with any point on its opposite side. This Student has visualized the decomposed shapes and recomposed it into a pentagon.

Influence of definition on its image

The results also indicate that the definition has influenced the image. There were some typical cases where this connection was clearly visible. For example, Student J defined rhombus as*: A four-sided closed figure with 90°angle is called rhombus. Each side is equal.*

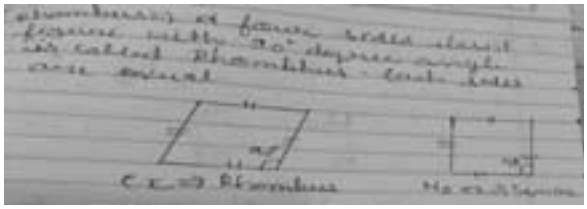


Figure 3: Sample of student's drawing of rhombus

Since the student defined the rhombus consisting of 90° angles, therefore he/she labels the obtuse angles in the figure as 90 degree angles. This connection between image and definition may work if students try to memorize the definition of figure that may lead to mix up properties of some other figure e.g. properties of square in this case are mixed up with properties of rhombus.

Influence of name of the figure on its definition and hence on image

The results also suggest that the name of the figure affects the ways in which students define the figure and also the kind of mental image they created of it. This was mainly observed while defining trapezium by 17% of class VIII students and 20% of class IX students. The trapezoid is called '*Samlamb*' in Hindi, and its literal meaning would be 'equal perpendicular'. Some interesting examples when asked to define a trapezoid student response were as follows: (translated from Hindi)

- *A figure in which one line is perpendicular to the second line.*
- *A line which stands on the other line is called 'Samlamb*

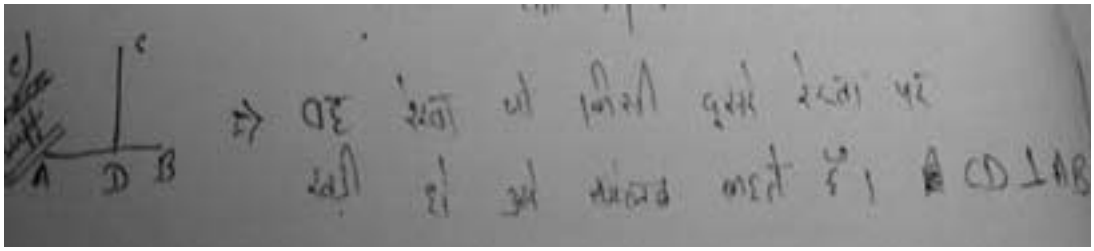


Figure 4: Sample of student's drawing of trapezium

Another example is:

- *Trapezium (Samlamb): A four-sided figure, with four equal sides and all sides make 90° angle, is called trapezium*

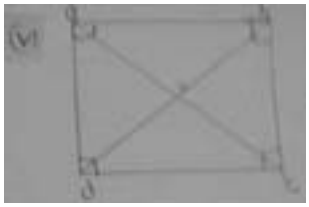


Figure 5: Sample of student's drawing of trapezium

CONCLUSIONS

Data of the study revealed that the students predominately used personal figural concept. There was a considerable gap between formal figural concept and personal figural concept of both classes VIII and IX. These findings are consistent with the finding of research done by Fujita and Jones (2007) ('kite' was not studied by them) and Kawasaki (1992) who studied teacher trainees' personal figural concepts of quadrilateral and found a significant gap between formal *figural* and *personal figural concept* of quadrilaterals. The study suggests interrelationship of quadrilaterals needs to be focussed more in school curriculum.

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